# USPAS 2019: Knoxville

# Transverse and Longitudinal Beam Dynamics Fundamentals


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# Lecture Outline

- Motivation
- Transverse Beam Dynamics
  - History (cyclotron, synchrotron, …)
  - Hill's Equation, Twiss Functions (Beta-Function, …)
  - Tune, Resonances, Emittance
  - Matrix Formalism
    - Basis for simulation codes
- Longitudinal Dynamics
  - Time of Flight, Synchrotron Oscillations
- Radiation
  - Damping/Excitation, Equilibrium Emittances

http://www2.als.lbl.gov/als_physics/csteier/uspas19/

# Motivation (recap from this Morning)

- This course deals in detail with measurements involving many areas of transverse (and longitudinal) single (and multiple) particle dynamics
- Lecture is reminder of concepts - somewhat consistent starting point
  - Will introduce lattice functions in two different ways (including the one usually used in lattice codes like AT which we use in he computer class).
- *Admittedly a packed lecture. Most of our class will be more practical and example based*
  - *and does not require full understand of this lecture/recap*
- *Disclaimer: Our class is storage ring biased.*
  - *Basic concepts and measurements are applicable to transfer lines and linacs, but details are different. If you have questions regarding lines, linacs, protons: You are welcome to ask at any time.*
  - *We have added some non ring examples and we can set aside more teaching/discussion time after some of the ring examples.*

# Transverse Beam-dynamics: Terminology

- Linear beam-dynamics determined by:
  - Dipoles
  - Quadrupoles (lenses)
  - Solenoids
  - RF-resonators
  - Linear approximation of synchrotron radiation
- Nonlinear (Wednesday/Thursday):
  - Sextupoles, higher multipoles, errors, insertion devices (undulators/wigglers), stochastic nature of SR, …
- Trajectory/Orbit – (more on closed orbit tomorrow)
  - Closed orbit: closed, periodic trajectory around a ring (closes after one turn in position and angle).
  - Particles that deviate from the closed orbit will oscillate about it (transverse: Betatron oscillations, longitudinal: Synchrotron Oscillations)

# Math Concepts used in this lecture

- Some differential equations
  - (Vectorized) Maxwell equations
  - Hill’s equation
    - Mostly qualitative
- Linear Algebra
  - Matrix multiplication, fixed points, orthogonalization

# Electromagnetic Fields

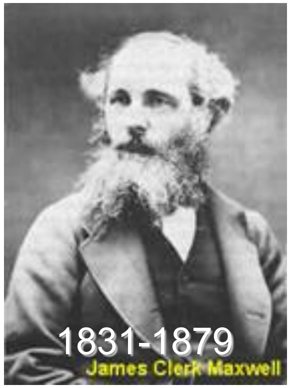
Maxwell Equations in vacuum (SI Units – differential form):

$$\nabla \cdot \overline{E} = \frac{\rho}{\varepsilon_0}$$ Coulomb's or Gauss' law for electricity

$$\nabla \times \overline{E} = -\frac{d\overline{B}}{dt}$$ Faraday's law

$$\nabla \cdot \overline{B} = 0$$ Gauss' law for magnetism

$$\nabla \times \overline{B} = \mu_0 \overline{J} + \mu_0 \varepsilon_0 \frac{d\overline{E}}{dt}$$ Ampere's law




Time variable magnetic fields are <u>always</u> associated with electric fields (and vice versa)

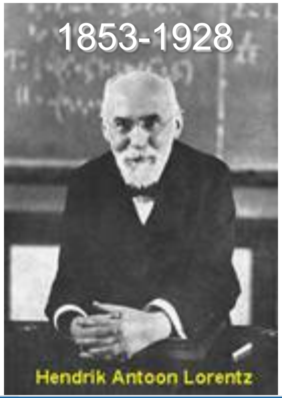
# Lorentz Equation

$$\bar{F} = q\left(\bar{E} + \bar{v} \times \bar{B}\right)$$

$$W = \int \bar{F} \cdot d\bar{l} = q\int \bar{E} \cdot d\bar{l} + q\int \left(\bar{v} \times \bar{B}\right) \cdot d\bar{l}$$

B fields can change the trajectory of a particle
But <u>cannot</u> do *work* and thus change its energy

$$\bar{F} = q\bar{E} \qquad W = q\int \bar{E} \cdot d\bar{l}$$



# Early Accelerators: Cyclotron

- In a cyclotron, charged particles
  - circulate in a strong magnetic field
  - are accelerated by electric fields in gaps
    - away from gaps particles are screened from electric field.
    - when particles enter the next gap, phase of time-varying voltage has changed 180 degrees so particles are again accelerated.
- Cyclotron condition:

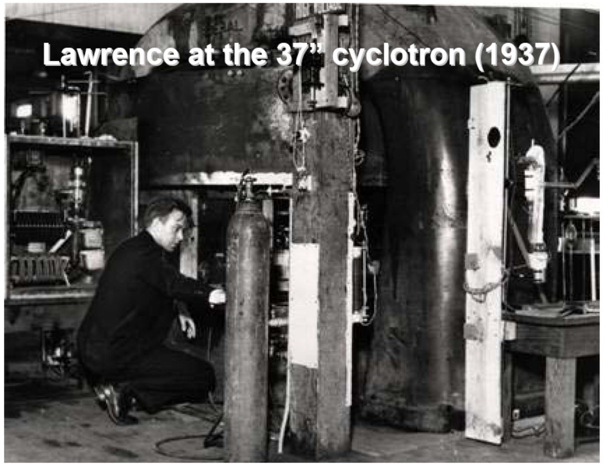
Centripetal force=Lorentz force$=e[\vec{E}+\vec{v}\times\vec{B}]$

$$\frac{mv^2}{\rho}=evB$$

$$\rho=\frac{mv}{eB}=\frac{p}{eB}$$

$$f=\frac{v}{2\pi\rho}=\frac{eB}{2\pi m}$$

The first cyclotron 4.5" diameter (1929).

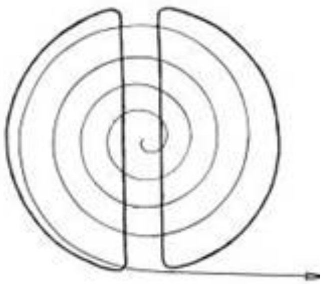



Lawrence's cyclotrons were foundation of LBNL

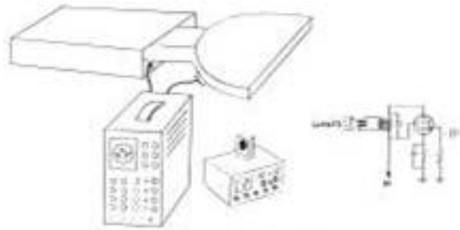
- Only works for non-relativistic particles

# The Cyclotron: Different Points of View

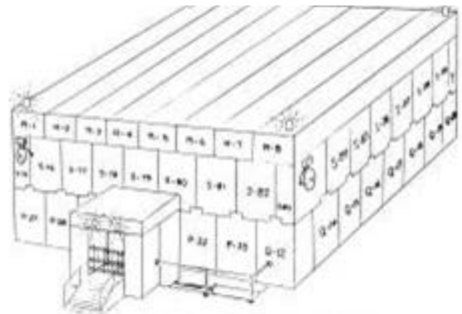
The Cyclotron, as seen by...



... the inventor

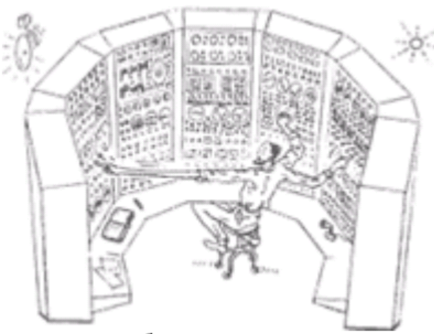


... the electrical engineer



... the health physicist

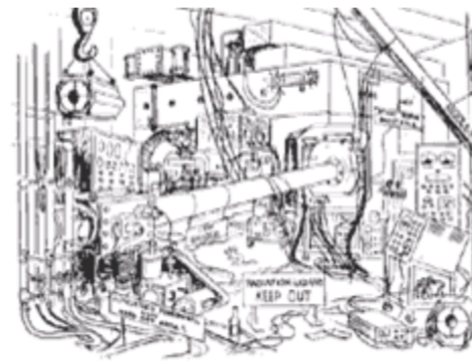
... the experimental physicist



...the operator

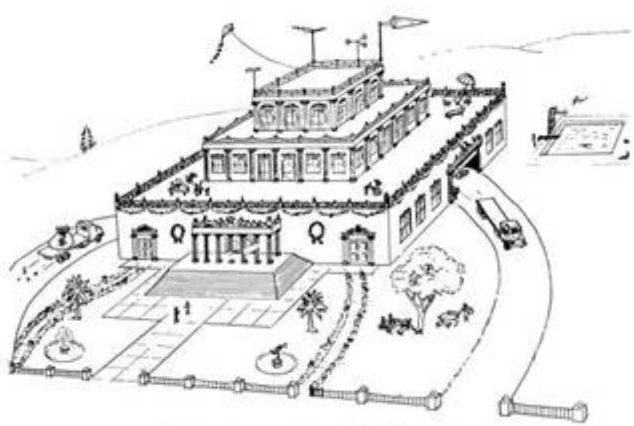
**From LBNL Image Library Collection**

**By Dave Judd and Ronn MacKenzie**

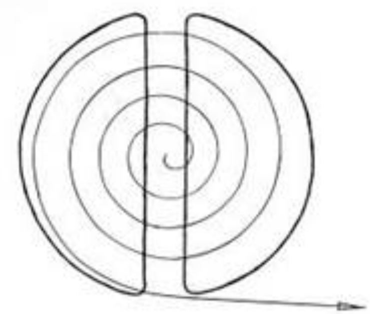


...the visitor

... the laboratory director



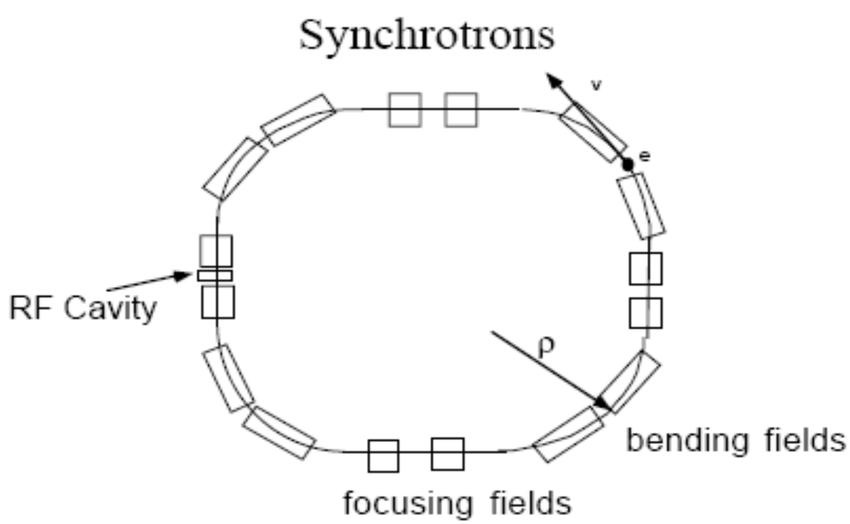
... the governmental funding agency



... the student

# Higher Energy Reach: Synchrotron (1945)

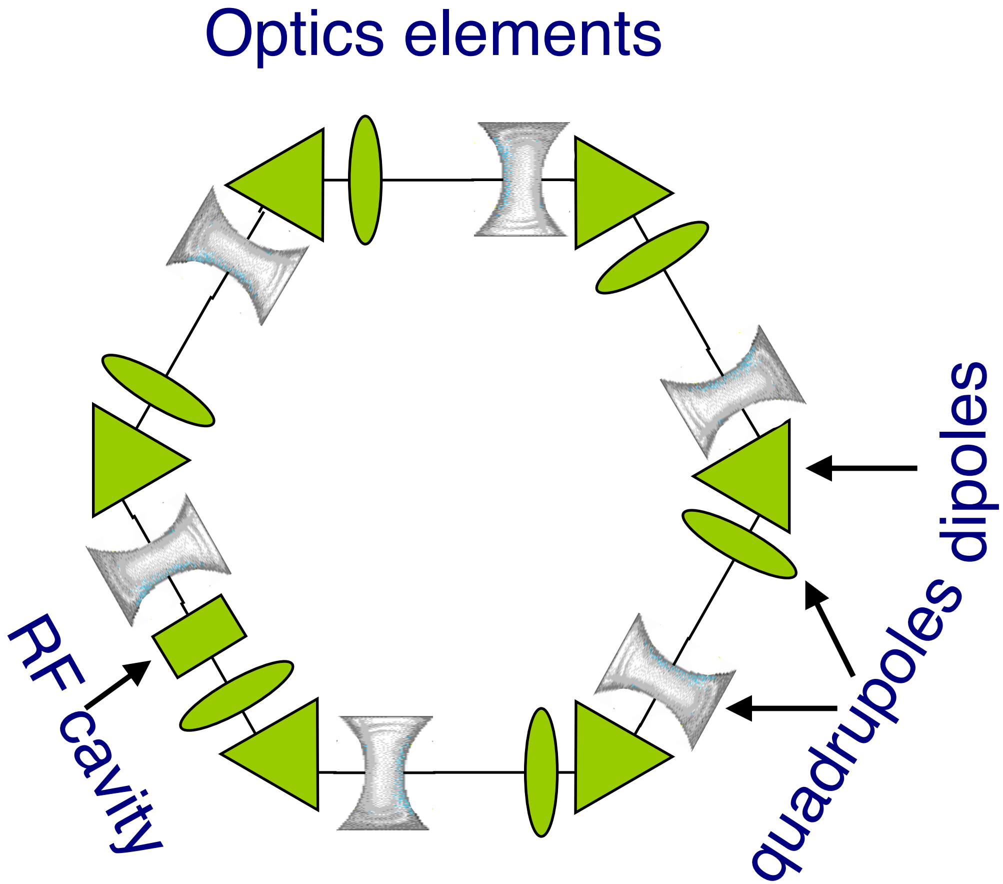
- Synchrotrons allowed extending beam energies much beyond cyclotrons, originally for elementary particle physics
- Synchrotron is a circular accelerator with discrete magnets along the beam path, which has one (or a few) electromagnetic resonant cavity to accelerate the particles. **A constant orbit is maintained during the acceleration.**
  - First ones were weak focusing (very large vacuum chambers and magnets)
  - Later strong focusing.
- Originally ramping/cycling, today often storage rings (many hours lifetime)

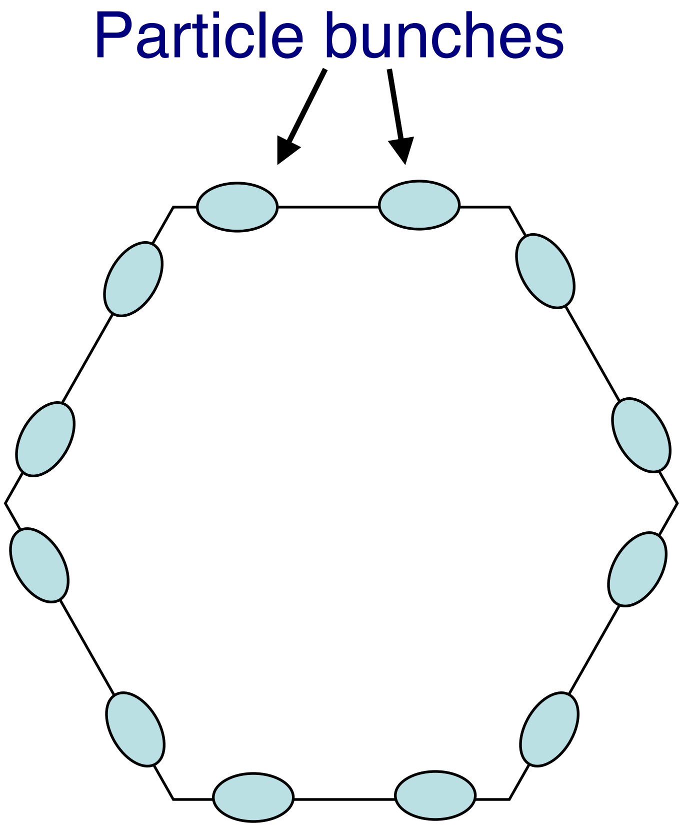



**The synchrotron concept seems to have been first proposed in 1943 by the Australian physicist <u>Mark Oliphant.</u>**

# RF acceleration -> beam is bunched

In particle storage rings, charged particles circulate around the ring in bunches for a large number of turns.



# Lorentz Force –> Equation of Motion

**The motion of each charged particle is determined by the electric and magnetic forces that it encounters as it orbits the ring:**

- **Lorentz Force**

$$F = ma = e(E + v \times B),$$

$m$ **is the relativistic mass of the particle,**

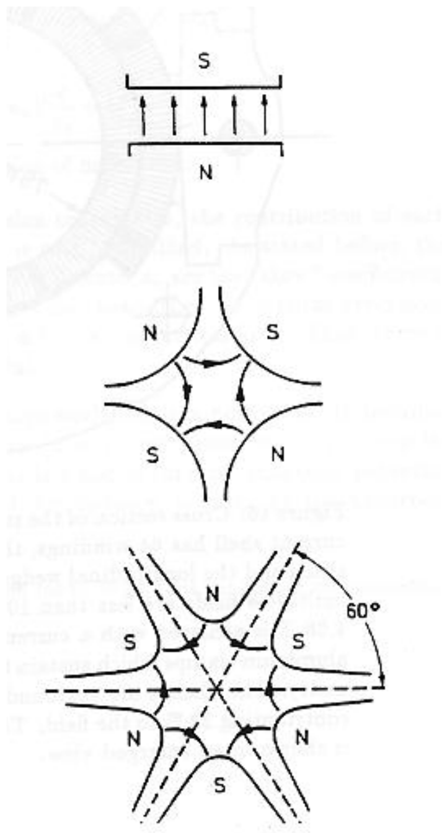
$e$ **is the charge of the particle,**

$v$ **is the velocity of the particle,**

$a$ **is the acceleration of the particle,**

$E$ **is the electric field and,**

$B$ **is the magnetic field.**




- **Typically acceleration with electrical fields, guidance with magnetic ones**

# Typical Magnet Types

There are several magnet types that are used in storage rings:

Dipoles → used for guiding

$B_x = 0$

$B_y = B_o$

Quadrupoles → used for focussing

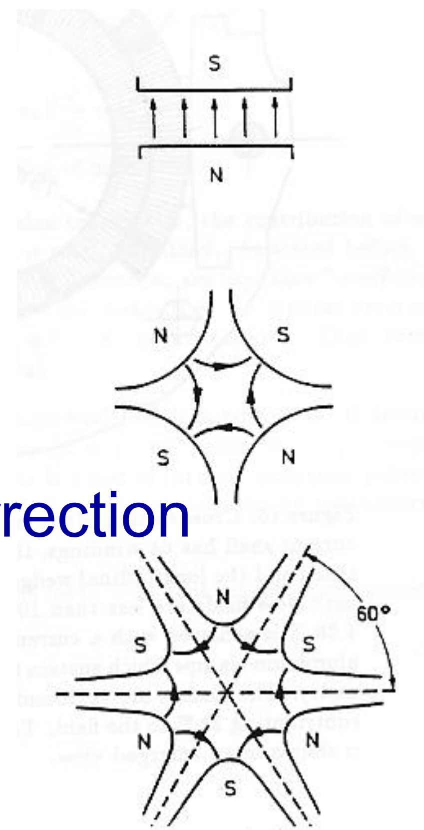
$B_x = Ky$

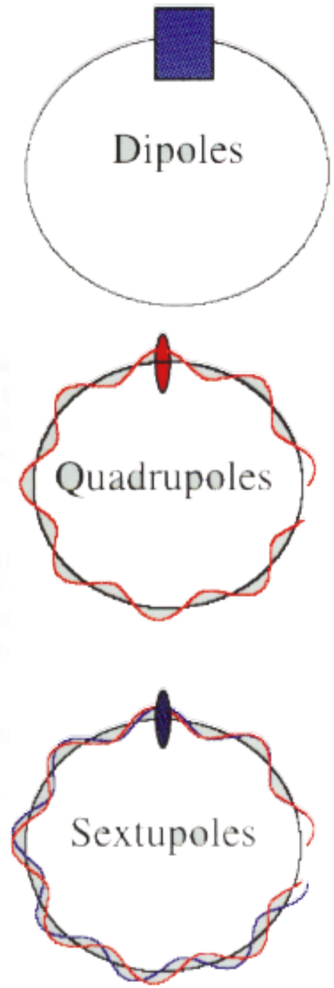
$B_y = -Kx$

Sextupoles → used for chromatic correction

$B_x = 2Sxy$

$B_y = S(x^2 – y^2)$

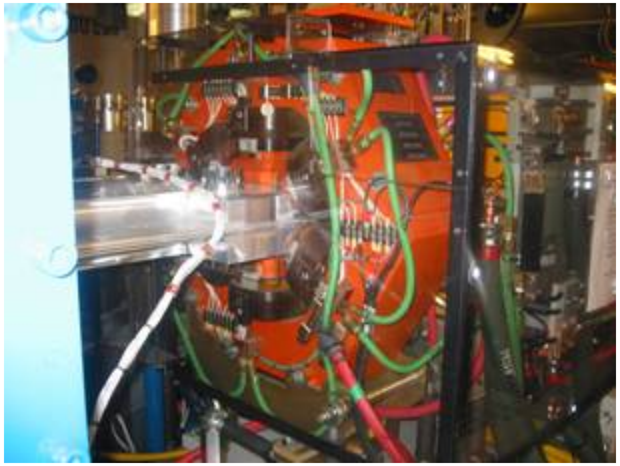


# Magnet Examples at the ALS in Berkeley

Dipoles

Quadrupoles



Sextupoles

# Differential Equation vs. Matrix Formalism

There are two approaches to describe the motion of particles in a storage ring

1. The traditional way in which one begins with Hill's equation, defines beta functions and dispersion, and how they are propagated, …

2. The way that our computer models do the calculations

I will begin with the first way (as a brief recap) but spend most of the time with the second approach

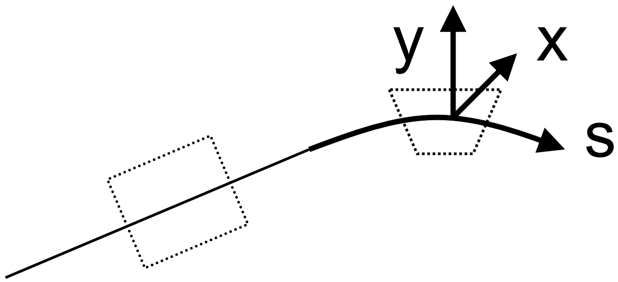
# Discussion – Coordinate Systems

- Coordinate system choice for accelerators
  - Why would one pick a 'non-standard' coordinate system
  - Options for coordinate systems
  - Other examples you might have encountered
  
  …

# Coordinate System

Change dependent variable from time to longitudinal position, *s*

Coordinate system used to describe the motion is usually locally Cartesian or cylindrical




Typically the coordinate system chosen is the one that allows the easiest field representation

# Hill's equation

This approach, using differential equations, provides some insights into concepts but is limited in usefulness for actual calculations

We begin with on-energy no coupling case. The beam is transversely focused by quadrupole magnets. The horizontal linear equation of motion is

$$\frac{d^2 x}{ds^2} = -k(s)x,$$

Looks (almost) like harmonic oscillator equation, except for s dependence of restoring force (k(s)

**where** $k = \frac{B_T}{(B\rho)a}$, **with**

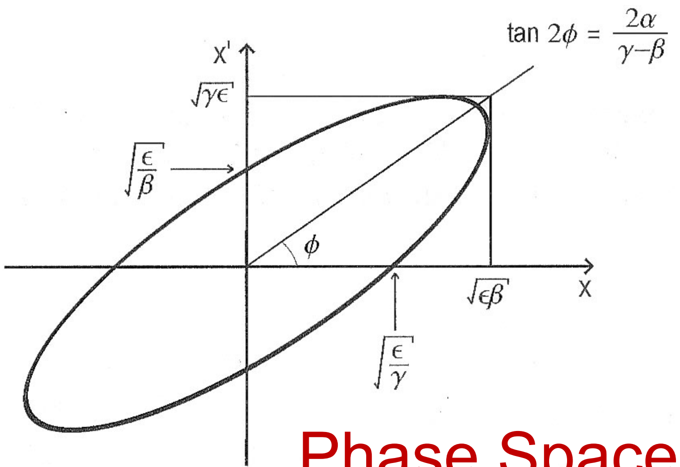
$B_T$ **being the pole tip field**

$a$ **the pole-tip radius, and**

$B\rho$**[T-m]** $\approx 3.356\, p$**[GeV/c]**

# Solutions of Hill's equation

The solution can be parameterized by a pseudo-harmonic oscillation of the form:




Phase Space

$$x_\beta(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos(\varphi(s)+\varphi_0)$$

$$x'_\beta(s) = -\sqrt{\varepsilon}\frac{\alpha}{\sqrt{\beta(s)}}\cos(\varphi(s)+\varphi_0) - \frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\sin(\varphi(s)+\varphi_0)$$

**where $\beta(s)$ is the beta function,** -> Size

**$\alpha(s)$ is the alpha function,** -> Divergence

**$\varphi_{x,y}(s)$ is the betatron phase, and**

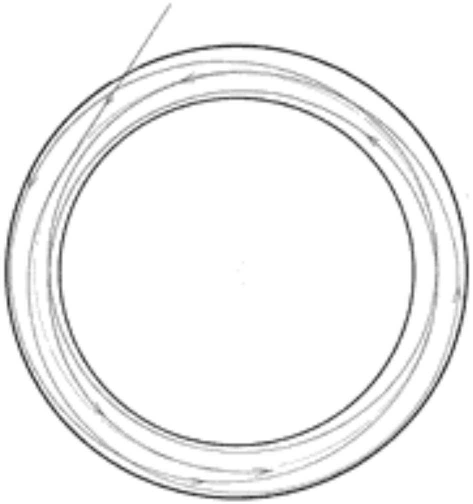
**$\varepsilon$ is an action variable** -> emittance, conserved

$$\varphi = \int_0^s \frac{ds}{\beta}$$

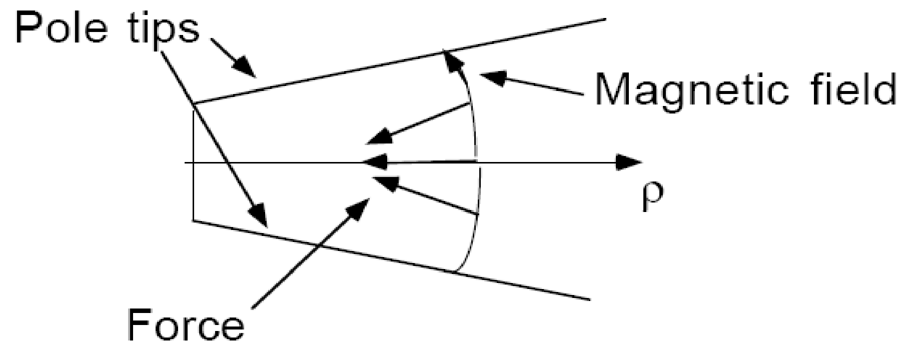
$$\alpha = -\frac{\beta'}{2},$$

$$\gamma = \frac{1+\alpha^2}{\beta}$$

# Weak Focusing



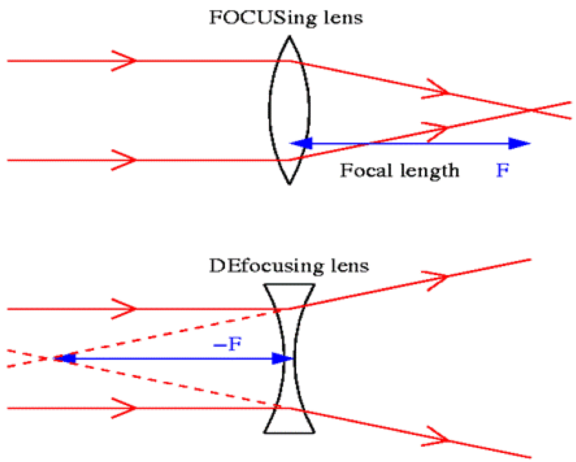
Weak focusing accelerator



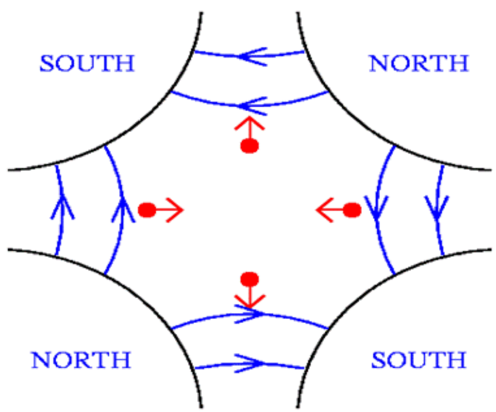

- Horizontally, homogenous dipole magnetic field ‘focuses’ - discuss
- Vertically, trajectories simply diverge in homogenous dipole field
- Introducing a field gradient (radially decreasing field) provides vertical focusing
  - but it reduces horizontal focusing
  - in cyclotron this causes particles to get out of sync with RF
- Weak focusing also causes large beta functions – i.e. big beams

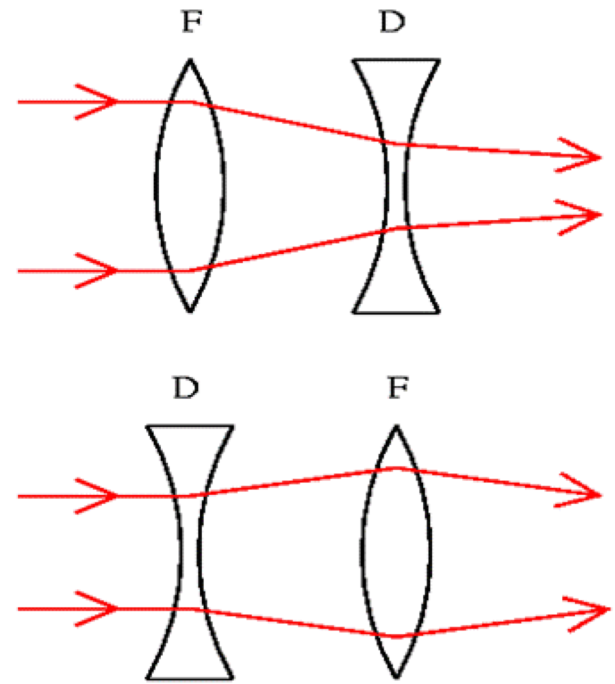
# Alternating Gradient Focusing

OPTICAL lens




MAGNETIC lens

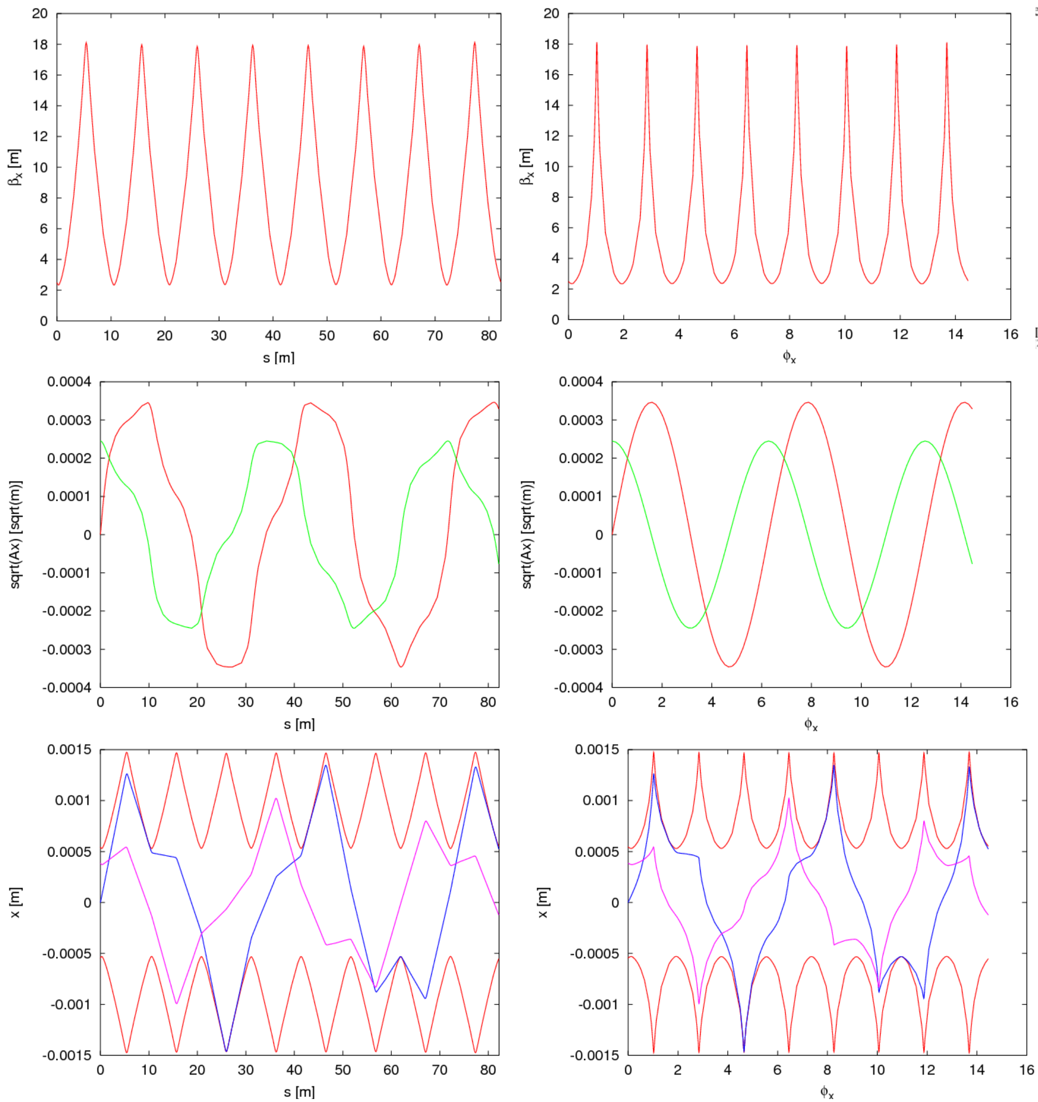


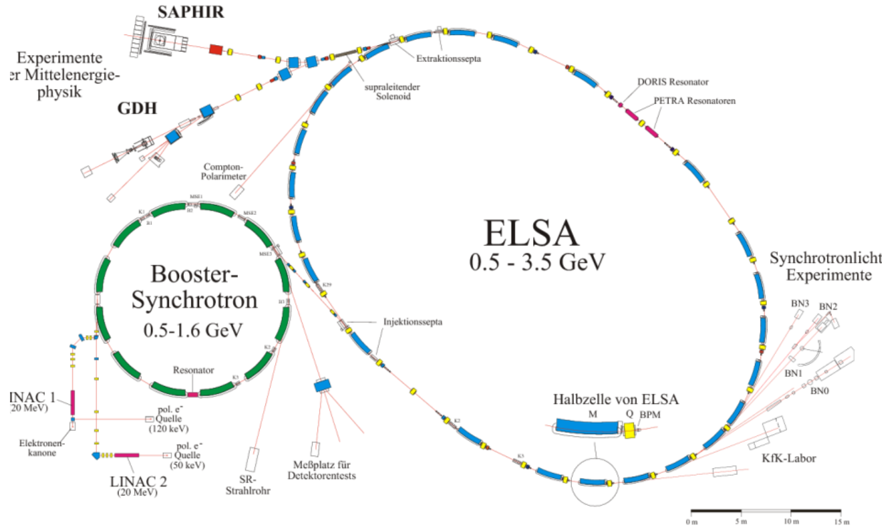




- Magnetic lenses (quadrupoles) cannot be focusing in both planes (Maxwell equations)
- Use alternating gradient / strong focusing
  - Reduces beta functions / beam sizes compared to weak focusing
- System of just two quadrupoles can be focusing in both planes

$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$ is positive for a large range of focal lengths and d=> net focusing both radially and vertically

# Example of Twiss functions and trajectories



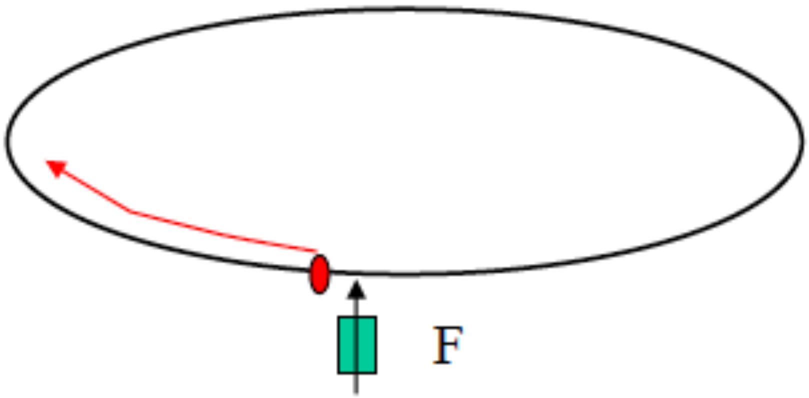

- ELSA (Electron Stretcher and Accelerator) in Bonn, where I did my PhD thesis, is example of simple FODO lattice
  - Beta Function highly periodic
- Trajectories in real space are piecewise straight (with deflections at quadrupoles)
- If one transforms normalizes with beta functions and phase advance, it looks like harmonic functions (sine/cosine)

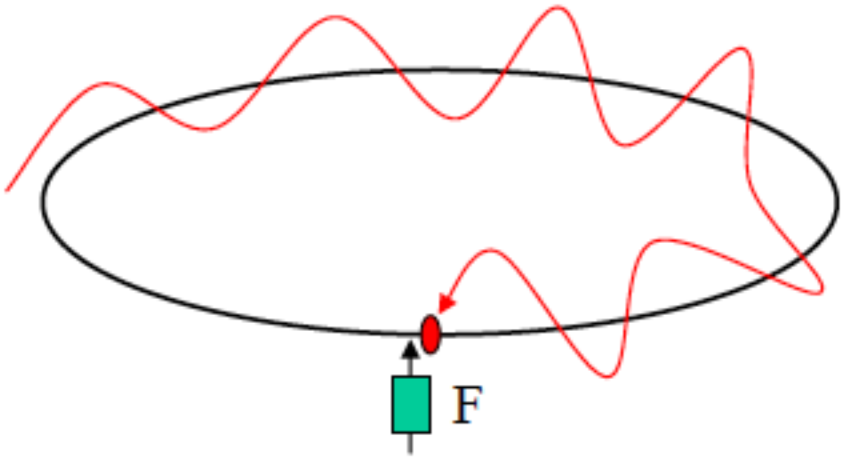
# Damped and driven harmonic oscillator – Resonances (will revisit Thursday)

- While Hill's equation is for a free oscillator, field errors in real accelerators provide driving term
- General solution for a driven oscillator is sum of
  - transient (the solution for damped harmonic oscillator, homogeneous ODE), depends on initial conditions
  - and a steady state (particular solution of the nonhomogenous ODE), independent of initial conditions; depends on driving frequency, driving force, restoring force, damping force
- Damped harmonic oscillator differential equation:

$$\frac{d^2u(t)}{dt^2} + \frac{\omega_0}{Q}\frac{du(t)}{dt} + \omega_0^2 u(t) = \frac{F}{m}\cos(\omega t)$$

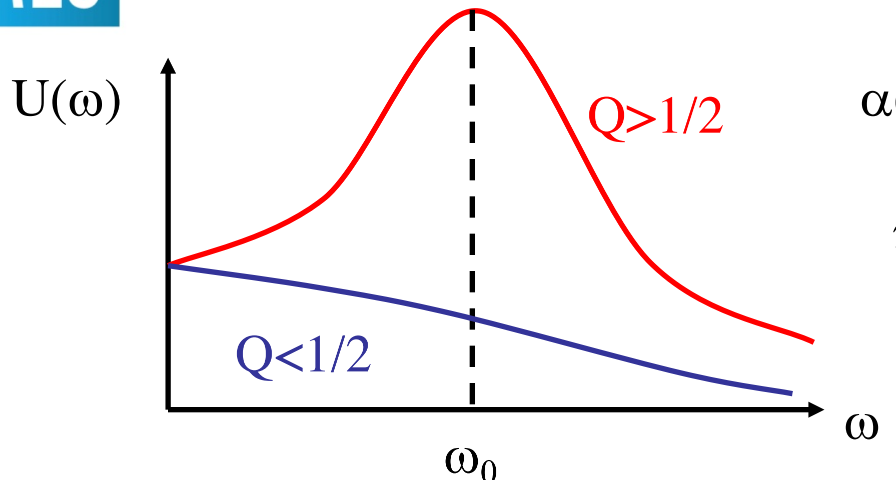
# Driven Oscillator – time dependent excitation vs. fixed in space

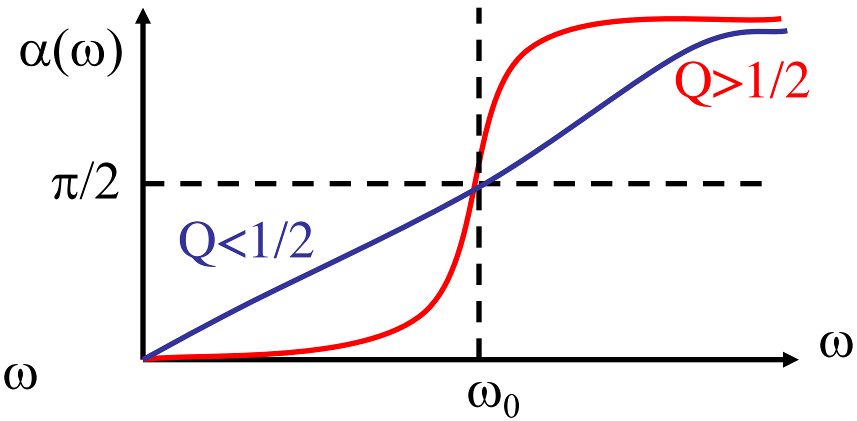



- Driven harmonic oscillator
  - periodic excitations
  - frequency of excitation determined by external source
- Betatron oscillations
  - Excitation due to field error, fixed in space (and usually not time dependent)
  - Excitation frequency is determined by oscillation frequency of beam particles
- Both result in similar driven resonances

# Resonance effect (more on Wed/Thurs)

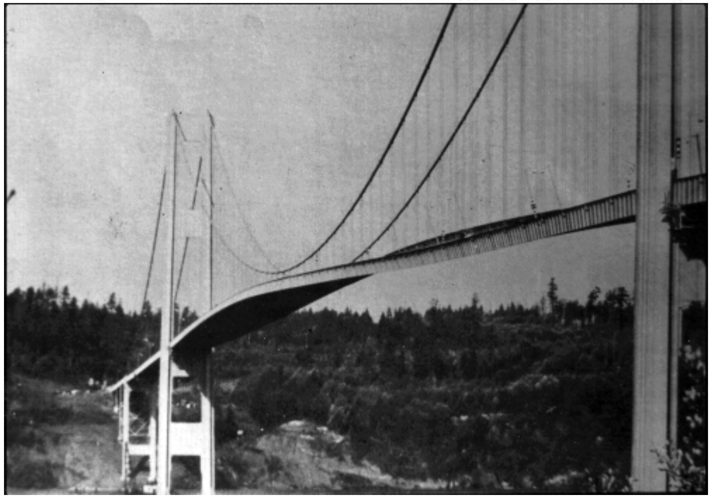



$$U(\omega) = \frac{U(0)}{\sqrt{(1-(\frac{\omega}{\omega_0})^2)^2 + (\frac{\omega}{Q\omega_0})^2}}$$

- Without or with weak damping a resonance condition occurs for $\omega = \omega_0$
- Infamous example:

Tacoma Narrow bridge 1940
Excitation at bridge eigenfrequencies (resonant modes) by strong wind



# 2nd Approach to calculate lattice functions, used by tracking codes

**Begin with equations of motion → Lorentz force**

↓

**Change dependent variable from time to longitudinal position**

↓

**Integrate particle trajectory around the ring and find the closed orbit**

↓

**Generate a map around the closed orbit**

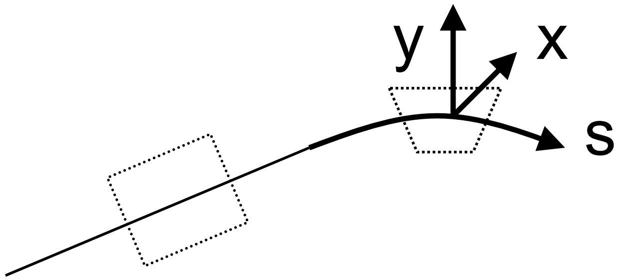
↓

**Analyze and track the map around the ring**

# Same as for Hill's equation: Coordinate System Choice

Change dependent variable from time to longitudinal position, *s*

Coordinate system used to describe the motion is usually locally Cartesian or cylindrical



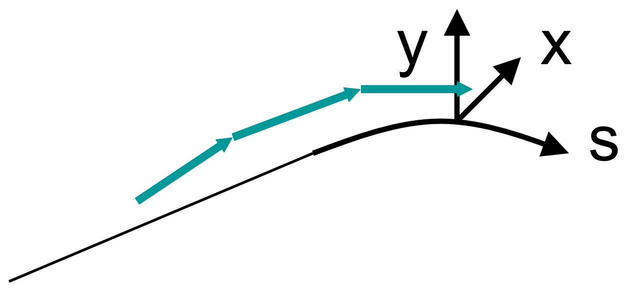

Typically the coordinate system chosen is the one that allows the easiest field representation

# Key Method: Integrate/Advance Element by Element

Integrate through the elements

Use the following coordinates*

$$x,\ x' = \frac{dx}{ds},\ y,\ y' = \frac{dy}{ds},\ \delta = \frac{\Delta p}{p},\ \tau = \frac{\Delta L}{L}$$

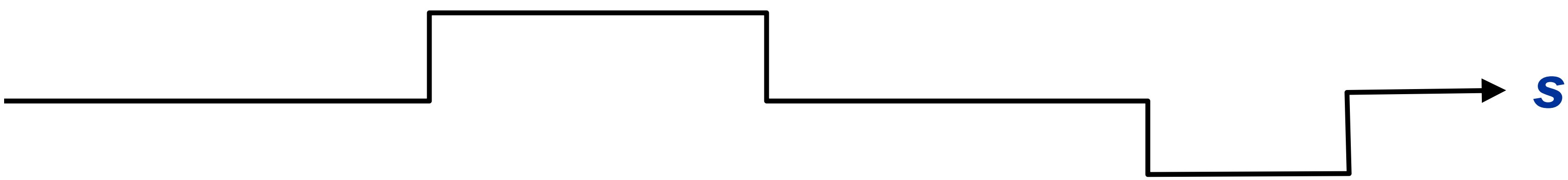



**Note sometimes one uses canonical momentum rather than x' and y'*

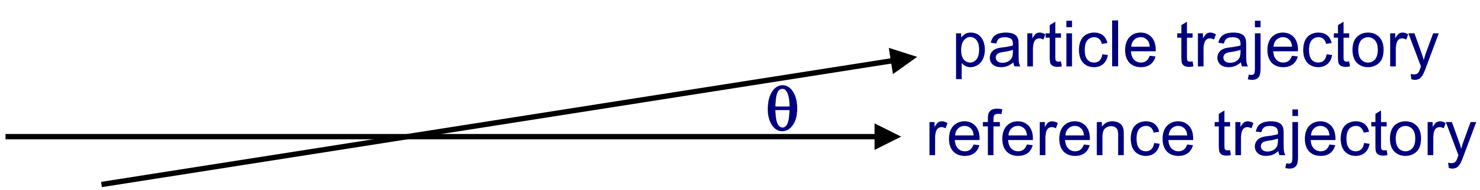
# Approximation

Everything up to now was general. No discussion of the field representation or the integrator. In many codes simplifications are made.

1. The velocity of the particle is the speed of light → $v = c$
2. The magnetic field is isomagnetic. Piecewise constant in $s$



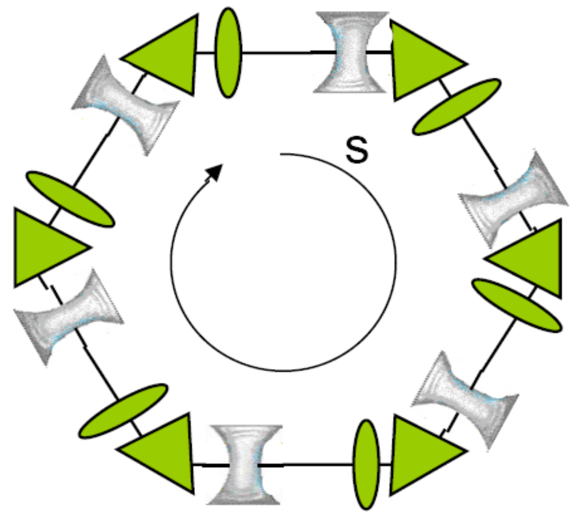

3. The angle of the particles with respect to the reference particle is small and can assume that $\theta = \tan\theta$



# Linear Algebra - Concatenation

- One can write the linear transformation from one point in the accelerator ($s_0$) to another one (s) as:

$$\begin{pmatrix} u \\ u' \end{pmatrix}_s = \mathcal{M}(s|s_0) \begin{pmatrix} u \\ u' \end{pmatrix}_{s_0} = \begin{pmatrix} C(s|s_0) & S(s|s_0) \\ C'(s|s_0) & S'(s|s_0) \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix}_{s_0}$$




- Note that

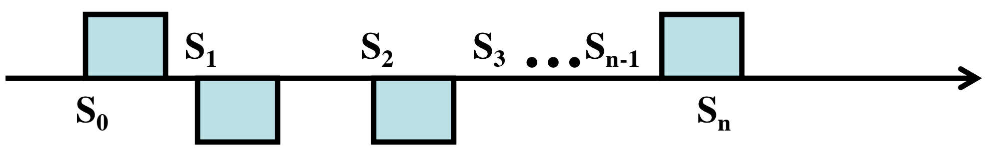
$$\det(\mathcal{M}(s|s_0)) = C(s|s_0)S'(s|s_0) - S(s|s_0)C'(s|s_0) = 1$$

which is always true for conservative systems

- Note also that $\mathcal{M}(s_0|s_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathcal{I}$

- The accelerator can be modeled by a series of matrix multiplications

$$\mathcal{M}(s_n|s_0) = \mathcal{M}(s_n|s_{n-1}) \ldots \mathcal{M}(s_3|s_2) \cdot \mathcal{M}(s_2|s_1) \cdot \mathcal{M}(s_1|s0)$$




from $s_0$ to $s_1$

from $s_0$ to $s_2$

from $s_0$ to $s_3$

from $s_0$ to $s_n$

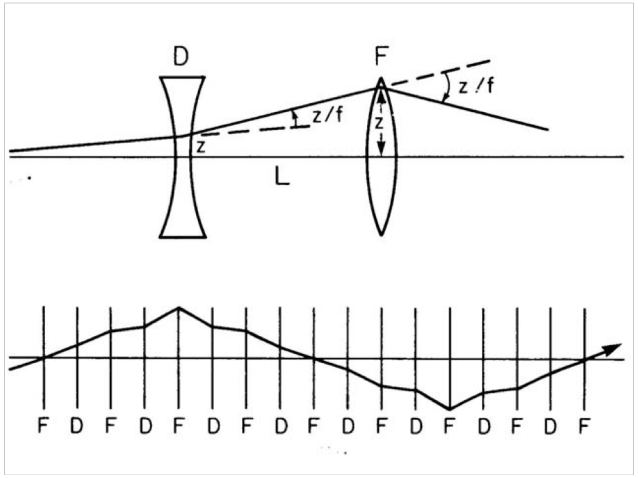
# Examples of transfer matrices

Drift of length $L$

$$R_{drift} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

**The matrix for a focusing quadrupole of gradient** $k = (\partial B / \partial x)/(B\rho)$ **and of length** $l_q$

$$R_{Quad} = \begin{pmatrix} \cos\phi & \sin\phi / \sqrt{|k|} \\ -\sqrt{|k|}\sin\phi & \cos\phi \end{pmatrix}$$




**The matrix for a zero length thin quadrupole** $K = |k|\, l_q$

$$R_{thin-lens} = \begin{pmatrix} 1 & 0 \\ -K & 1 \end{pmatrix}$$

# Computer Code Example: AT

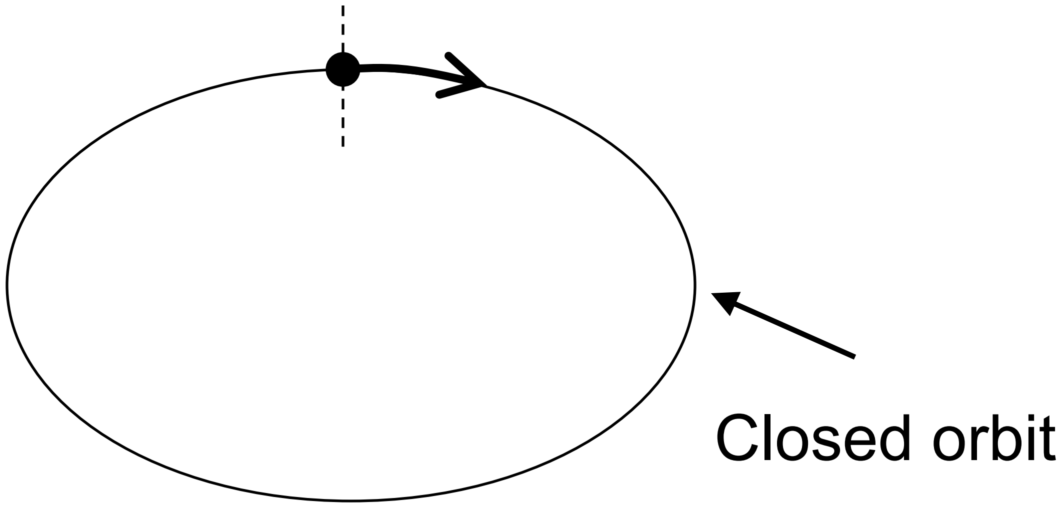
```
…
              g  = fabs(K)/(1+r [4]);
              t  = sqrt(g);
              lt = L*t;
 if(K>0) {   /* Horizontal  */
              MHD = cos(lt);
              M12 = sin(lt)/t;
              M21 = -M12*g;
              /* Vertical */
              MVD = cosh(lt);
              M34 = sinh(lt)/t;
              M43 = M34*g;          }
else …
```

$$R_{Quad} = \begin{pmatrix} \cos\phi & \sin\phi/\sqrt{|k|} \\ -\sqrt{|k|}\sin\phi & \cos\phi \end{pmatrix}$$

QuadLinearPass.c

# The Closed Orbit

A closed orbit is defined as an orbit on which a particle circulates around the ring arriving with the same position and momentum that it began.




In every working story ring there exists at least one closed orbit.

# One-turn Map R - Computation

A one-turn map, *R*, maps a set of initial coordinates of a particle to the final coordinates, one-turn later.

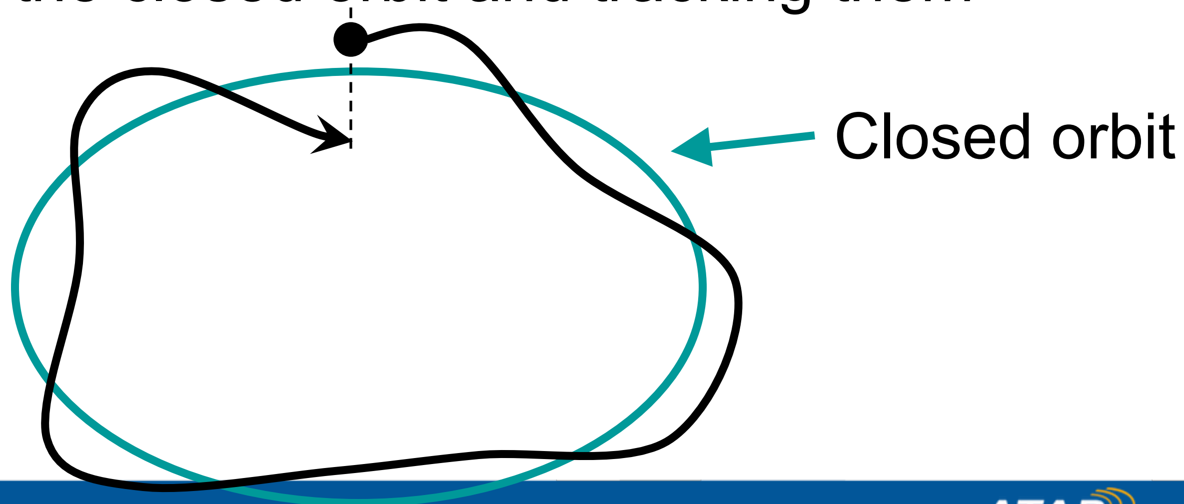
$$x_f = x_i + \frac{dx_f}{dx_i}\left(x_i - x_{i,co}\right) + \frac{dx_f}{dx'_i}\left(x'_i - x'_{i,co}\right) + \ldots$$

R11 R12

$$x'_f = x'_i + \frac{dx'_f}{dx_i}\left(x_i - x_{i,co}\right) + \frac{dx'_f}{dx'_i}\left(x'_i - x'_{i,co}\right) + \ldots$$

R21 R22

The map can be calculated by taking orbits that have a slight deviation from the closed orbit and tracking them around the ring.



# Beta-functions and tunes from 1-turn map

**The one turn matrix (the first order term of the map) can be written**

$$R_{one-turn} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \cos\varphi + \alpha\sin\varphi & \beta\sin\varphi \\ -\gamma\sin\phi & \cos\varphi - \alpha\sin\varphi \end{pmatrix}$$

**Where $\alpha$, $\beta$, $\gamma$ are called the Twiss parameters**

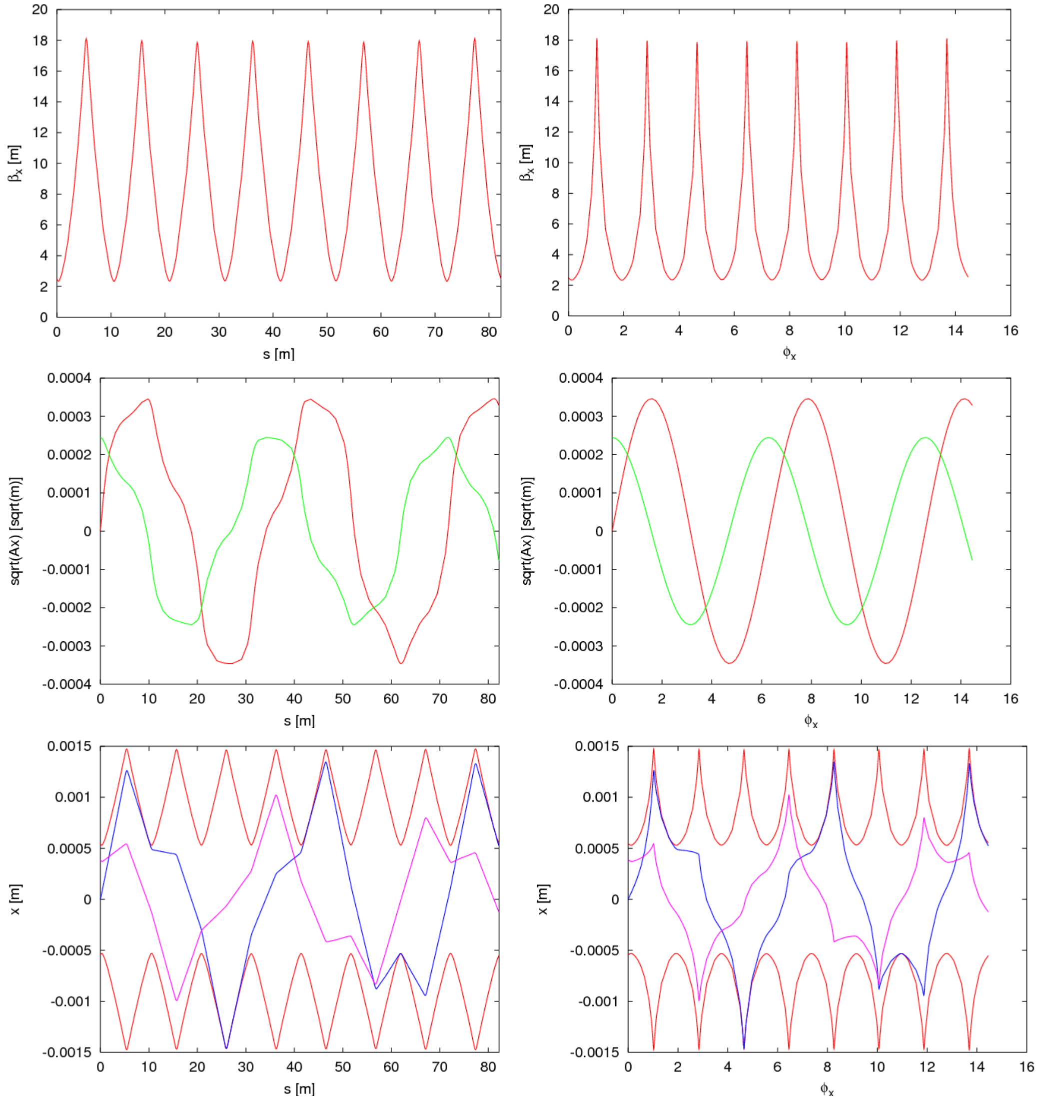
$$\alpha = -\frac{\beta'}{2}, \qquad \gamma = \frac{1+\alpha^2}{\beta}$$

**and the betatron tune, $\nu = \phi/(2*\pi)$**

***For long term stability $\phi$ is real ➔***

***$|TR(R)| = |2\cos\phi| < 2$***

# Recap: Example of Twiss functions and trajectories

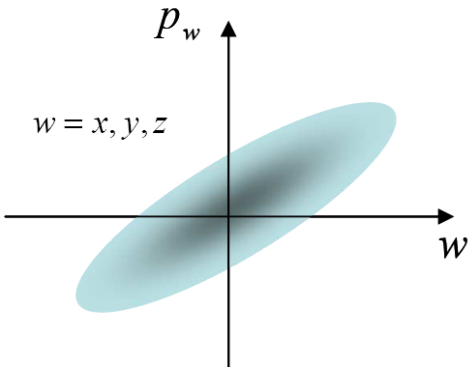



- The twiss functions and trajectories I showed earlier for ELSA were calculated with a tracking code
- I.e. trajectories are integrated piecewise
- Lattice functions are then calculated from one turn map

# Beam Emittance

- Consider the decoupled case and use the {*w*, *w*} plane where *w* can be either *x* or *y*:
  - The emittance is the phase space area occupied by the system of particles, divided by π

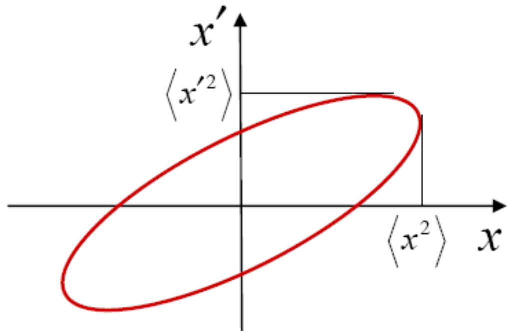
$$\varepsilon_w = \frac{A_{ww'}}{\pi} \qquad w = x, y$$




- *x*′ and *y*′ are conjugate to *x* and *y* when $B_z = 0$ and in absence of acceleration. In this case, we can immediately apply the Liouville theorem:
  - For such a system the emittance is an invariant of the motion.
- This specific case is very common in accelerators:
  - For most of the elements in a beam transferline, such as dipoles, quadrupoles, sextupoles, …, the above conditions apply and the emittance is conserved.
- Practical emittance example: In the transverse coordinates it is the product of the size (cross section) and the divergence of a beam (at beam waists).

# Emittance (2)

- Emittance can also be defined as a statistical quantity (beam is composed of finite number of particles)




$$\varepsilon_{geometric,rms} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$

- Using the twiss functions and the emittance, the beam envelopes (size, divergence, …) can be calculated at any place around the ring

$$\Sigma^x_{beam} = \varepsilon_x \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

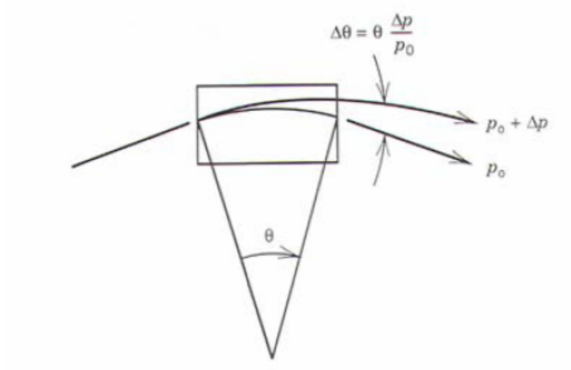
$$\Sigma^x_{beam,f} = R_{x,i-f} \Sigma^x_{beam,i} R^T_{x,i-f}$$

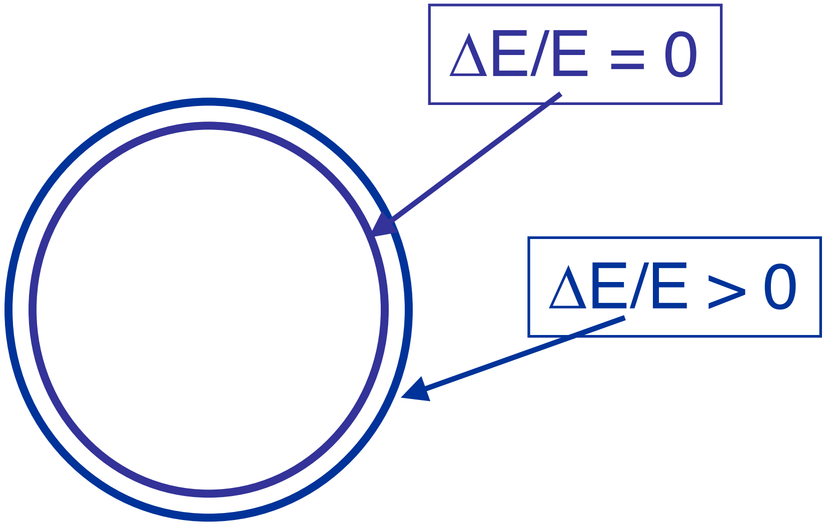
R are the linear transfer maps

# Off-Energy: Dispersion

- Assume that the energy is fixed → no cavity or damping
- Find the closed orbit for a particle with slightly different energy than the nominal particle.
- The Dispersion, *D*, is the change in closed orbit normalized by the energy difference.

$$x = D_x \frac{\Delta E}{E}$$

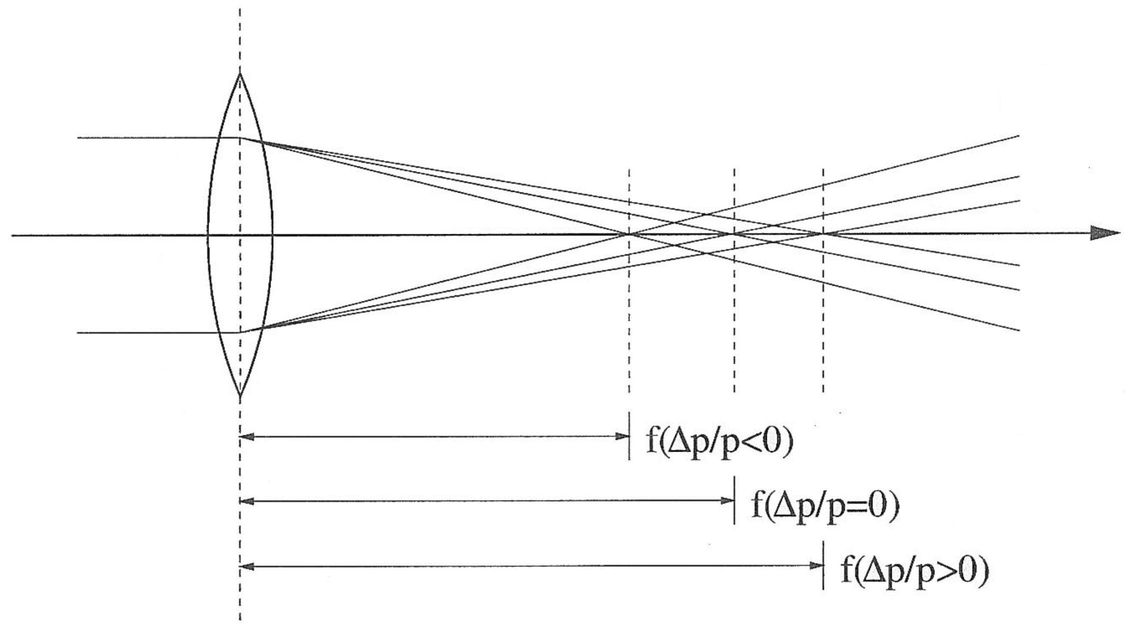



$$\begin{pmatrix} x \\ x' \\ \delta \end{pmatrix}_f = \begin{pmatrix} C & S & D_x \\ C' & S' & D'_x \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \delta \end{pmatrix}_i$$

# Off energy: Chromatic Aberration

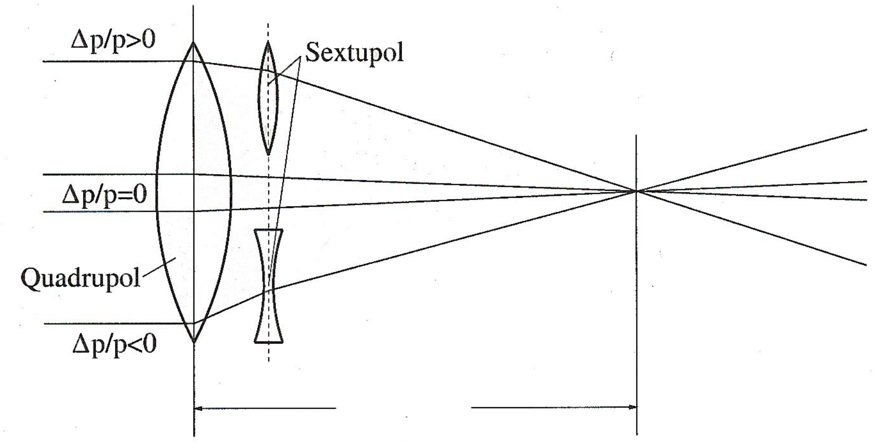
Focal length of the lens is dependent upon energy




Larger energy particles have longer focal lengths -> Chromaticity

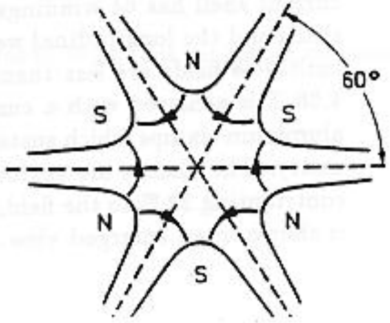
# Chromatic Aberration Correction

By including dispersion and sextupoles it is possible to compensate (to first order) for chromatic aberrations



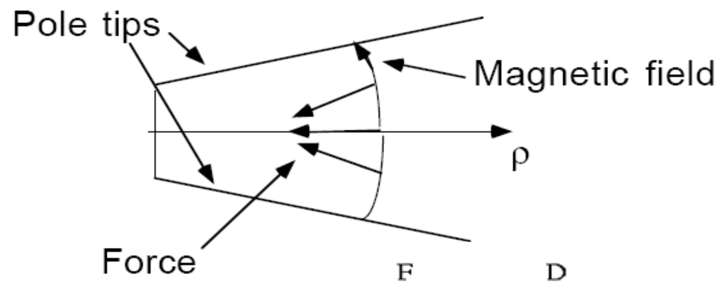

The sextupole gives a position dependent Quadrupole

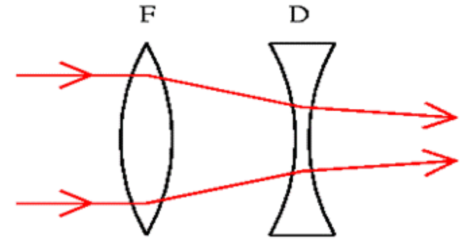
$$B_x = 2Sxy$$
$$B_y = S(x^2 - y^2)$$

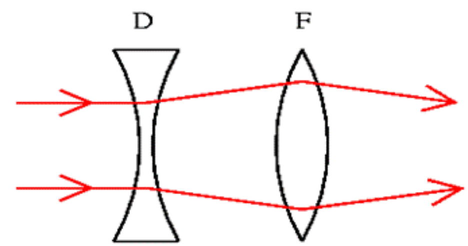


# Recap: Transverse Dynamics Concepts

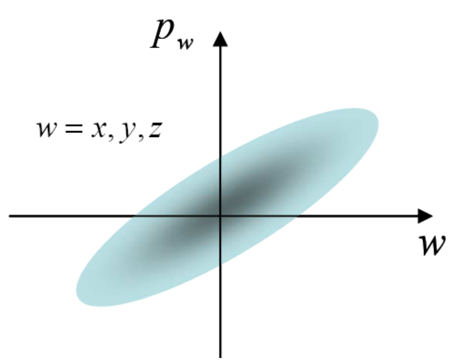
- Concepts introduced:
  - Weak Focusing
  - Alternating Gradient Focusing
  - Differential equation treatment (Hill's equation)
  - Resonances (driven harmonic oscillator)
  - Linear Algebra (matrix) treatment
  - Emittance - Liouville

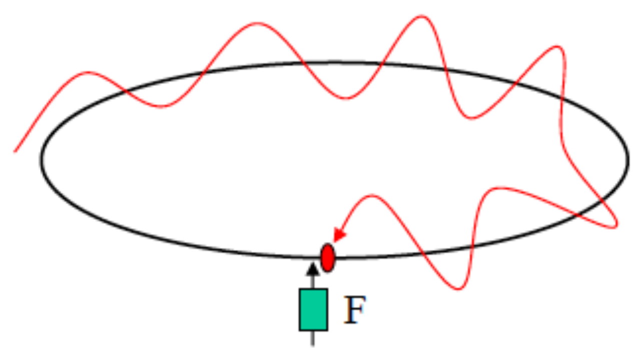



$$\frac{d^2x}{ds^2} = -k(s)x$$



# Longitudinal Dynamics in Storage Rings

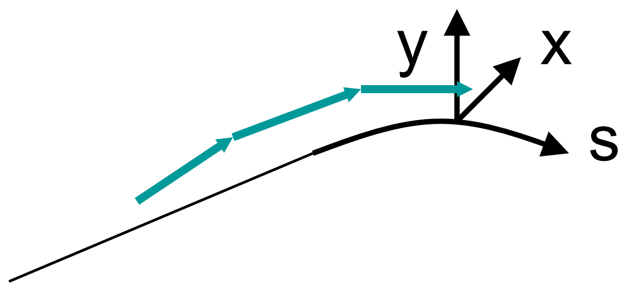
- Important case of acceleration:
  – RF fields used to accelerate particles
  – periodic accelerator (synchrotron or storage ring)
- Similar to transverse dynamics, the motion is (quasi) periodic
  – Important difference: Oscillations are slow compared to revolution period, therefore we do not need beta function formalism
- In addition to velocity term ($1/\gamma^2$), have to take path length into account
  – In general, higher energy particles tend to take wider turns, i.e. they need longer, opposite to the situation at low energies, where higher energy particles are faster

# Recap: Integrate – this time longitudinally

Integrate through the elements – in energy deviation and time

Use the following coordinates

$$x,\ x' = \frac{dx}{ds},\ y,\ y' = \frac{dy}{ds},\ \delta = \frac{\Delta p}{p},\ \tau = \frac{\Delta L}{L}$$




Relevant effects in longitudinal plane: velocity, path length, energy gain/loss (rf cavities, synchrotron radiation, …)

# Examples of 6D Transfer Matrix

Drift

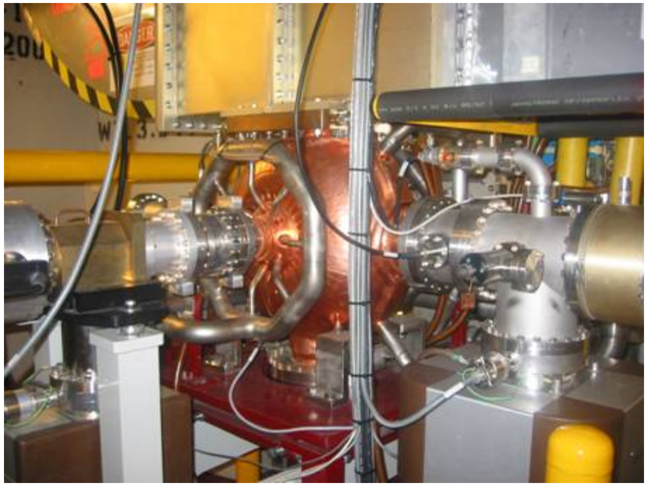
$$\begin{pmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & L & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\beta^2\gamma^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

thin RF cavity

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\omega\frac{e\hat{V}}{pc}\cos\phi & 1 \end{pmatrix}$$

coordinate vector

$$\begin{pmatrix} x \\ x' \\ y \\ y' \\ ct \\ \frac{\Delta p}{p} \end{pmatrix}$$



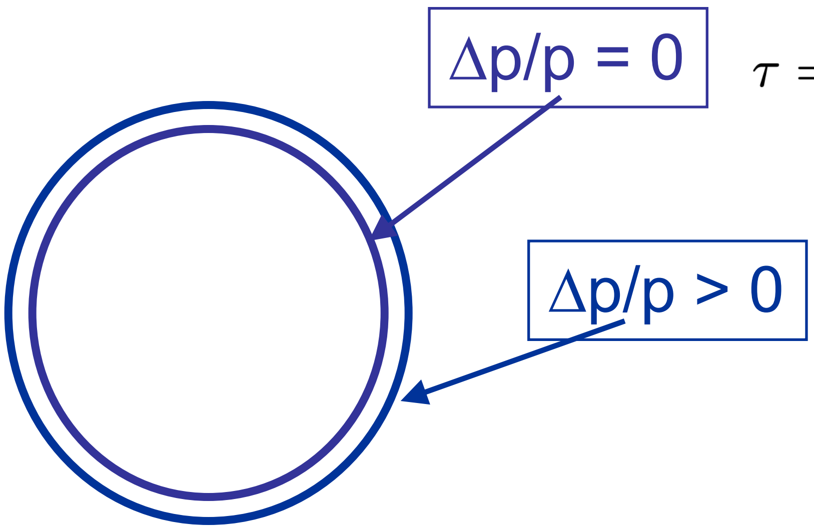
ALS RF cavity

Need to add path length effect in dipole – However, the exact one is too complex, so only explain effects on next few slides

# Dependence on revolution time on energy – momentum compaction

Again, assume that the energy is fixed → no cavity or damping

- Find the closed orbit for a particle with slightly different momentum
- Dispersion is the difference in closed orbit between them normalized by the relative momentum difference
- Momentum compaction factor relates the change in total closed orbit length to the momentum difference
  - From geometry one can find that it is the integral of the dispersion over the bending radius




$$\tau = \frac{\Delta L}{L} = -\left(\frac{1}{\gamma^2} - \alpha_c\right)\frac{\Delta p}{p} = -\eta_c\frac{\Delta p}{p} \approx \alpha_c\frac{\Delta p}{p}$$

v=c

$$\alpha_c = \int_0^{L_0} \frac{D_x}{\rho} ds$$

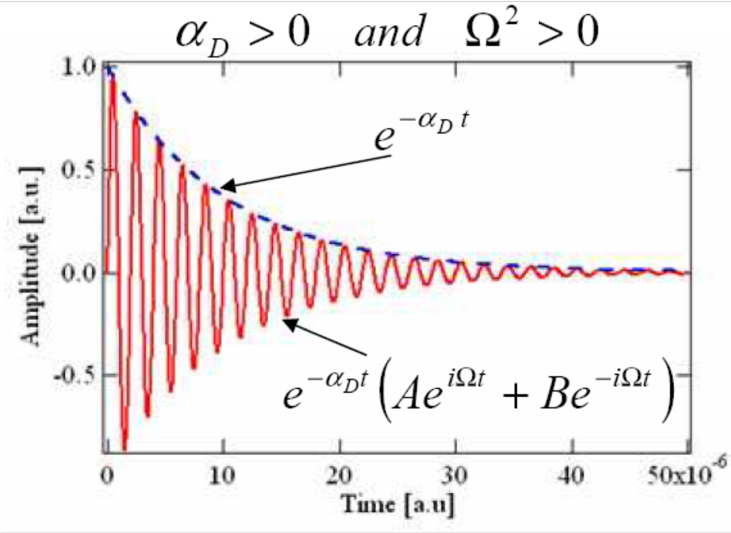
# Last Differential Equation of the Day: Synchrotron Oscillations

With the following definition for the frequency and neglecting damping,

$$\Omega^2 = \eta_c \frac{1}{p_0} \frac{q}{T_0} \frac{dV}{d\tau}\bigg|_{\tau_0}$$

we can write a simple harmonic oscillator like differential equation for synchrotron oscillations:

$$\frac{d^2\tau}{dt^2} + \Omega^2 \tau = 0$$




Solution are again harmonic functions:

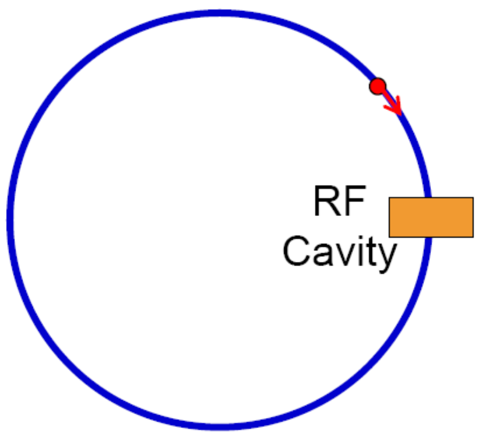
$$\tau(t) = Ae^{i\Omega t} + Be^{i\Omega t}$$

Or with Damping:

$$\tau(t) = e^{-\alpha_D t}\left(Ae^{i\Omega t} + Be^{i\Omega t}\right)$$

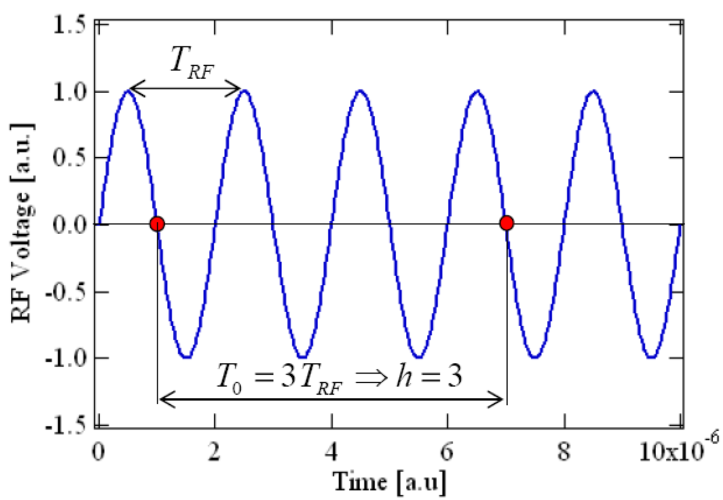
# Synchronicity/Harmonic Number

- The total length of the closed orbit is determined by the RF frequency, even if the physical circumference of the ring changes:




$$V_{RF}(t) = \hat{V}\sin(\omega_{RF} t)$$

$$T_0 = \frac{L_0}{\beta c} \qquad T_{RF} = \frac{1}{f_{RF}} = \frac{2\pi}{\omega_{RF}}$$




$$T_0 = hT_{RF} \Rightarrow f_0 = \frac{f_{RF}}{h}$$

**Synchronicity Condition**

The integer $h$ is called the *harmonic number*

The harmonic number sets the number of buckets (or possible bunches)

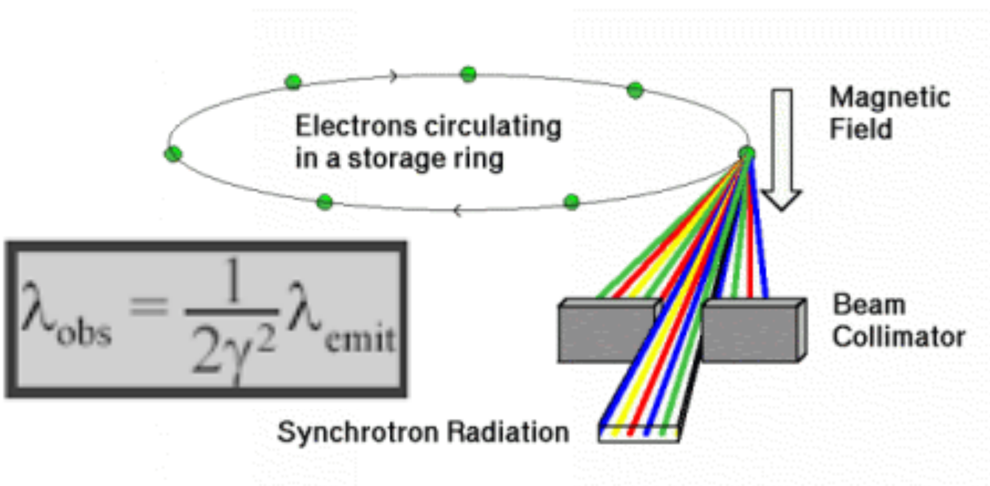
# Summary Longitudinal Dynamics

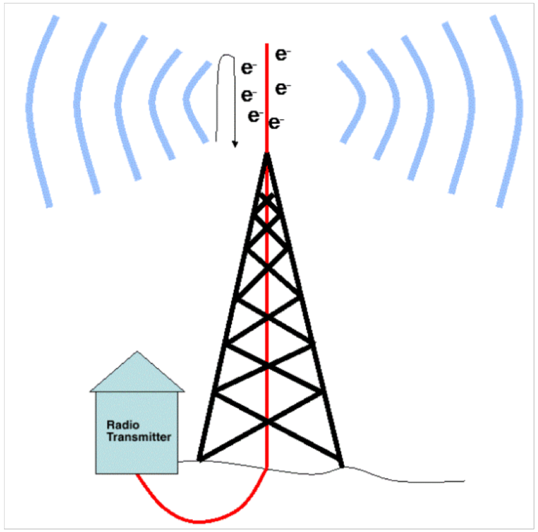
- Velocity variation often negligible (relativistic)
- Instead Path length difference very important
- Matrix treatment similar to transverse case
  - Can track simultaneously in 6D

# What is Synchrotron Radiation?

- Synchrotron radiation is electromagnetic radiation emitted when charged particles are radially accelerated (move on a curved path).

Electrons accelerating by running up and down in a radio antenna emit radio waves (long wavelength electromagnetic waves)



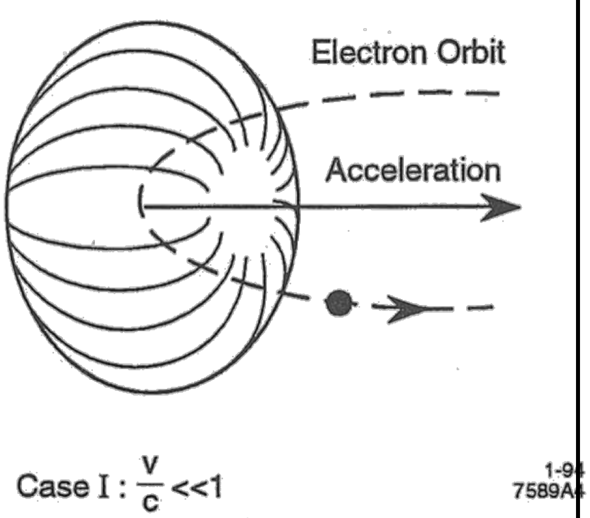




Both cases are due to the same fundamental principle:

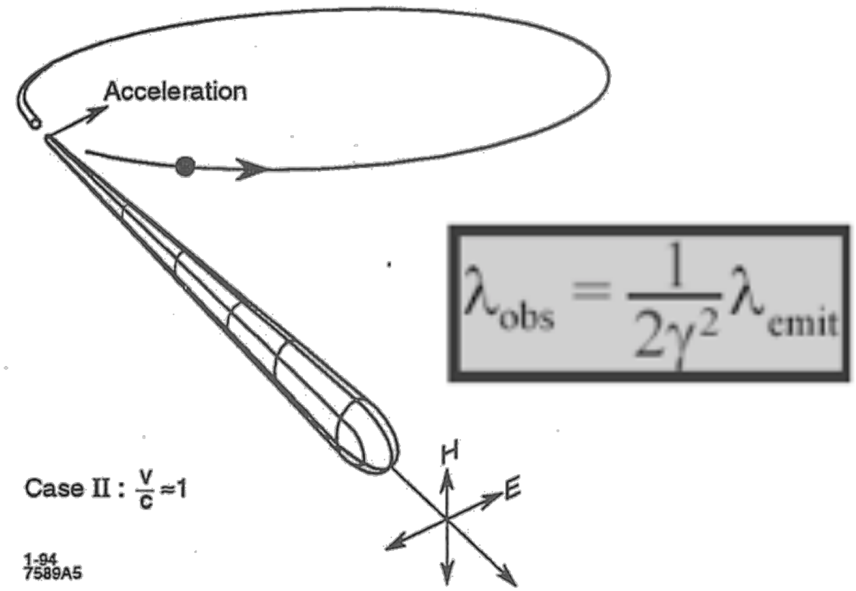
*Charged particles radiate when accelerated*.

# Synchrotron Radiation – Longitudinal + Transverse

- Radiated power increases at higher velocities
- Radiation becomes more focused at higher velocities




At low electron velocity (non-relativistic case) the radiation is emitted in a non-directional pattern




When the electron velocity approaches the velocity of light, the emission pattern is folded sharply forward. Also the radiated power goes up dramatically

# Radiation damping

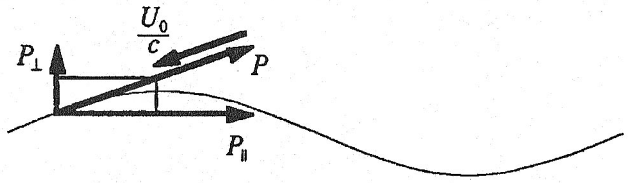
Energy damping:

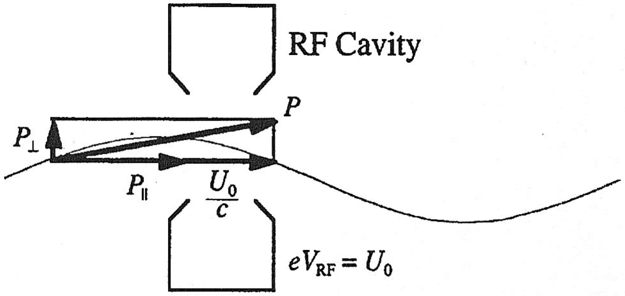
Larger energy particles lose more energy

$$P_{SR} = \frac{2}{3}\alpha\hbar c^2 \frac{\gamma^4}{\rho^2}$$

Transverse damping:

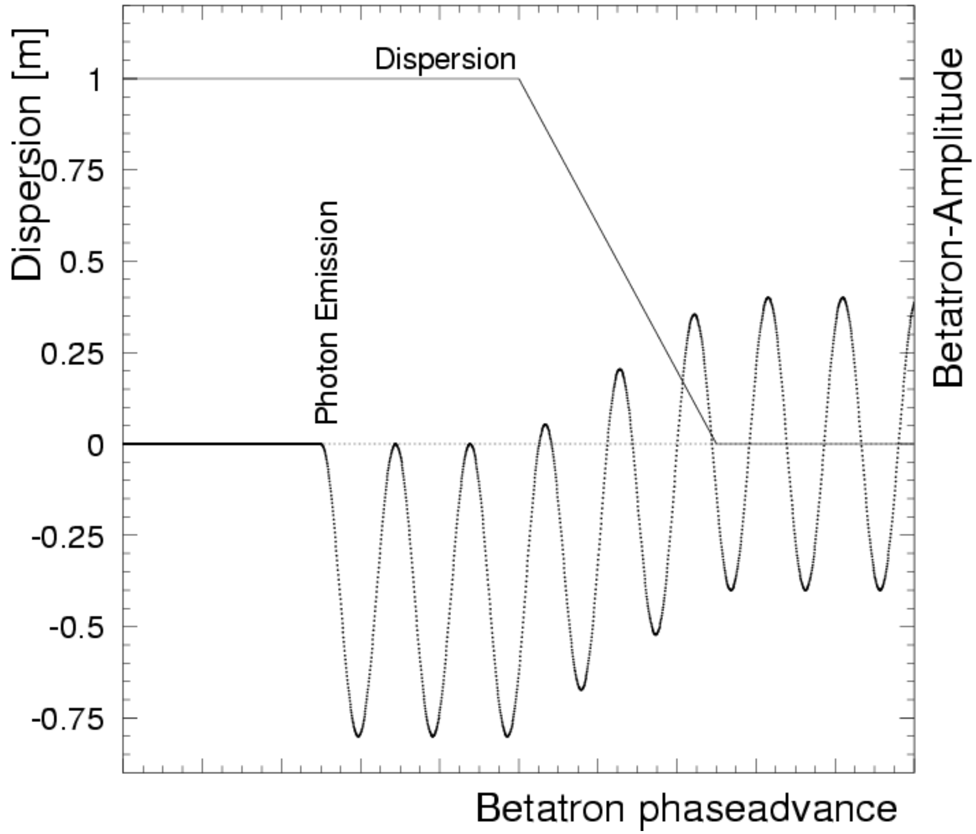
Energy loss is in the direction of motion while the restoration in the s direction



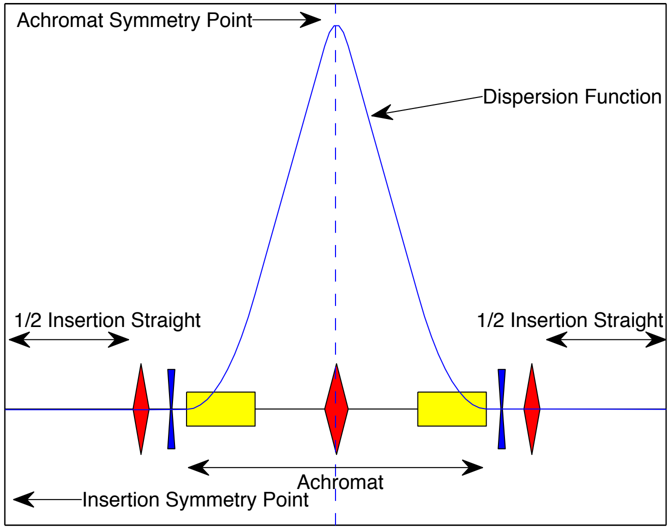
# Quantum Excitation - Transversely

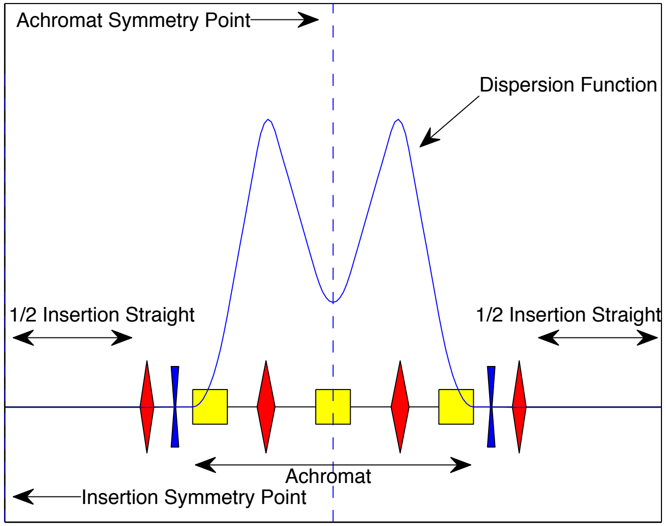
- Particle changes its energy in a region of dispersion
  - This induces increase in transverse oscillations.
- The balance with radiation damping gives the equilibrium emittances.

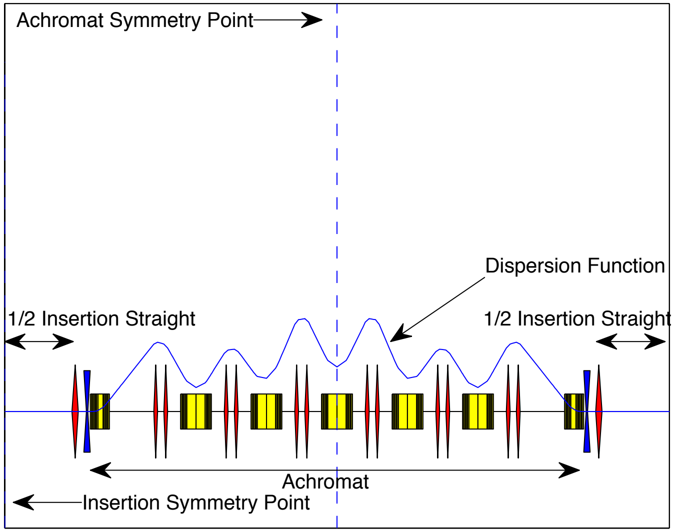



Dispersion can be minimized in lattice design

# Example: Low Emittance Lattices




- Early 3rd generation SR sources all used double/triple bend achromats
- Later optimization included detuning from achromatic condition
- New designs employ multi bend achromats
- To minimize emittance, stronger focusing is necessary
  - Strength limit of magnets
  - Chromaticity / Dynamic Aperture

# Comment: Wide Time Scales for Particle Dynamics in Rings

- We have discussed the motion of a particle in an accelerator for all 6 phase space dimensions
  - 4 transverse dimensions and 2 longitudinal ones
- An important effect is that the time scales for different phenomena are quite different:
  - Damping: several ms for electrons, ~ infinity for heavier particles
  - Betatron oscillations: ~ tens of ns
  - Synchrotron oscillations: ~ tens of μs
  - Revolution period: ~ hundreds of ns to μs

# Summary

- Recap of basic beam dynamics
  - Accelerator History
  - Transverse Focusing
  - Tune, Resonances
  - Emittance, Liouville
  - Acceleration, Longitudinal Dynamics in Rings
  - Synchrotron Radiation
    - Damping / Excitation

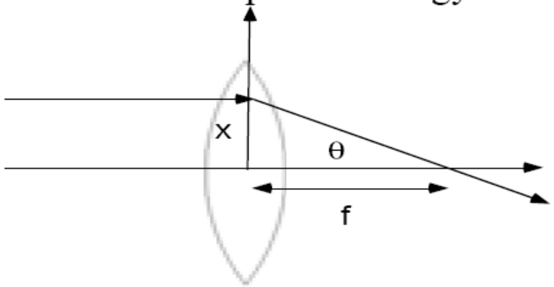
# List of Literature/Text Books

- Particle Accelerator Physics I (2nd edition, 1998), by Helmut Wiedemann, Springer
  - Or at a more advanced level: Particle Accelerator Physics II, H. Wiedemann, Springer (nonlinear dynamics, etc.)
- D.A. Edwards and M.J. Syphers, An Introduction to the Physics of High Energy Accelerators, John Wiley & Sons (1993)
- Accelerator Physics, S.Y. Lee, World Scientific, Singapore, 1999 (ISBN 9810237103)
- Many nice proceedings of CERN accelerator schools can be found at http://cas.web.cern.ch/cas/CAS_Proceedings.html , for the purpose of this class especially
  - CERN 94-01 v1 + v2
  - CERN 95-06 v1 + v2 (Advanced Class)
  - CERN 98-04 (Synchrotron Radiation+Free Electron Lasers)
- Accelerators and Nobel Laureates" by Sven Kullander which can be viewed at: https://www.nobelprize.org/prizes/themes/accelerators-and-nobel-laureates/

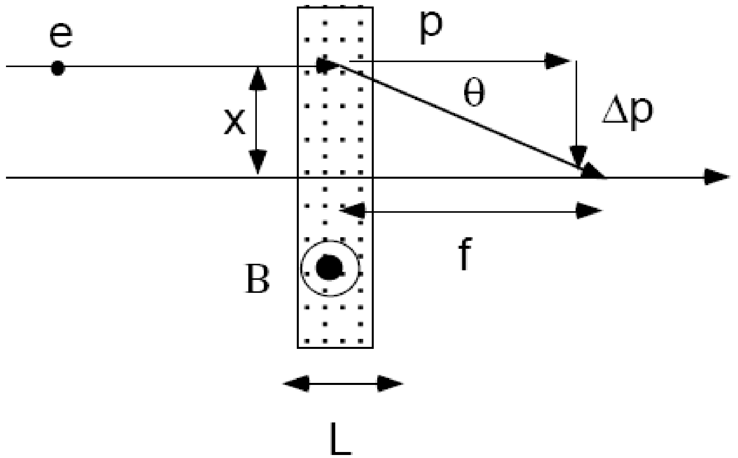
# Backup Slides

# Magnetic lenses: Quadrupoles

Magnetic focusing fields:
Optical analogy: Thin lens, focal length $f$




$$\theta = \frac{x}{f}$$

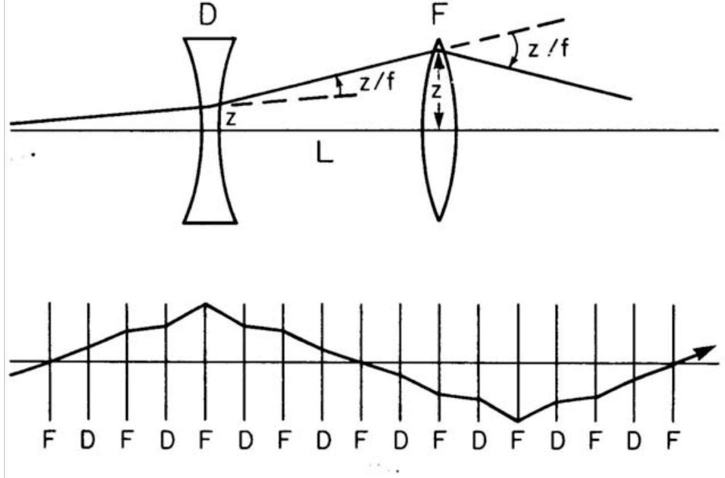



## Thin lens representation

$$\begin{pmatrix} 1 & L & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{f} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{f} & 1 \end{pmatrix} \quad \begin{pmatrix} x(s) \\ x'(s) \\ y(s) \\ y'(s) \end{pmatrix}$$

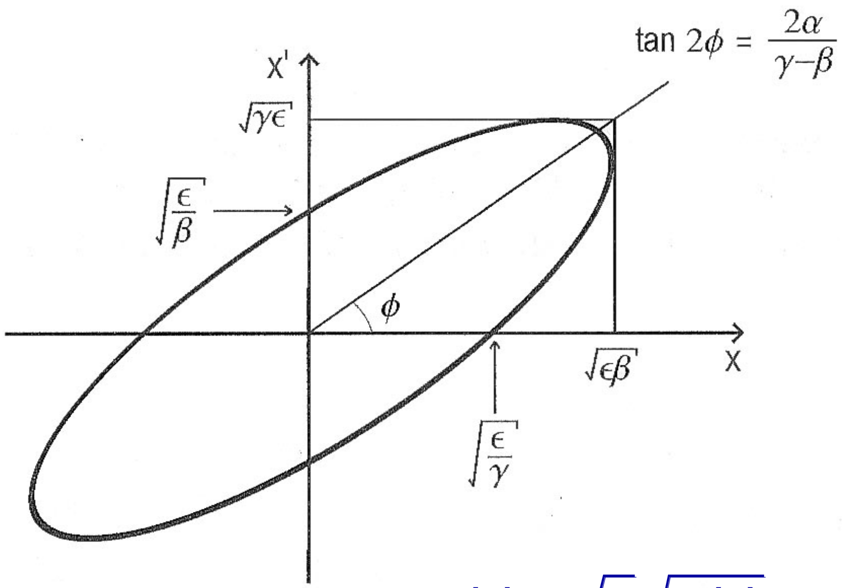
Drift: Thin lens:

## FODO cell



# Beam Ellipse

In an linear uncoupled machine the turn-by-turn positions and angles of the particle motion will lie on an ellipse




**Area of the ellipse, $\varepsilon$:**

$$\varepsilon = \gamma x^2 + 2\alpha x x' + \beta x'^2$$

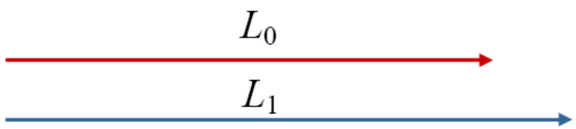
$$x_\beta(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos(\varphi(s)+\varphi_0)$$

$$x'_\beta(s) = -\sqrt{\varepsilon}\frac{\alpha}{\sqrt{\beta(s)}}\cos(\varphi(s)+\varphi_0) - \frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\sin(\varphi(s)+\varphi_0)$$

# Ballistic time-of-flight

- Consider two particles with different momentum on parallel trajectories:

$$p_1 = p_0 + \Delta p$$




- At a given time *t*:

$$L_1 = (\beta_0 + \Delta\beta)ct \qquad L_0 = \beta_0 ct \qquad \boxed{\Rightarrow \frac{\Delta L}{L_0} = \frac{L_1 - L_0}{L_0} = \frac{\Delta\beta}{\beta_0}}$$

- But:

$$p = \beta\gamma m_0 c \;\Rightarrow\; \Delta p = m_0 c\,\Delta(\beta\gamma) = m_0 c \gamma^3 \Delta\beta$$

$$\Rightarrow \frac{\Delta p}{p_0} = \gamma^2 \frac{\Delta\beta}{\beta} \qquad \boxed{\frac{\Delta L}{L_0} = \frac{1}{\gamma^2}\frac{\Delta p}{p_0}}$$

- The ballistic path length dependence on momentum is important everywhere, not just in bending magnets.
- Higher momentum particles are faster, i.e. precede the ones with lower momentum.
- The effect vanishes for relativistic particles.

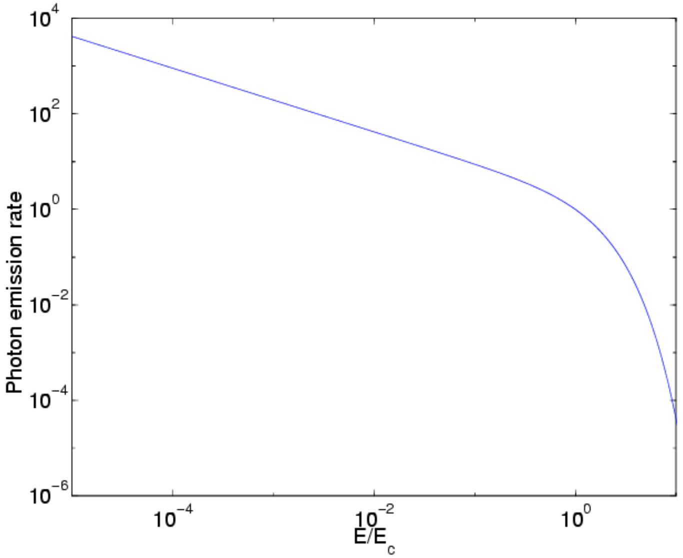
# Radiation

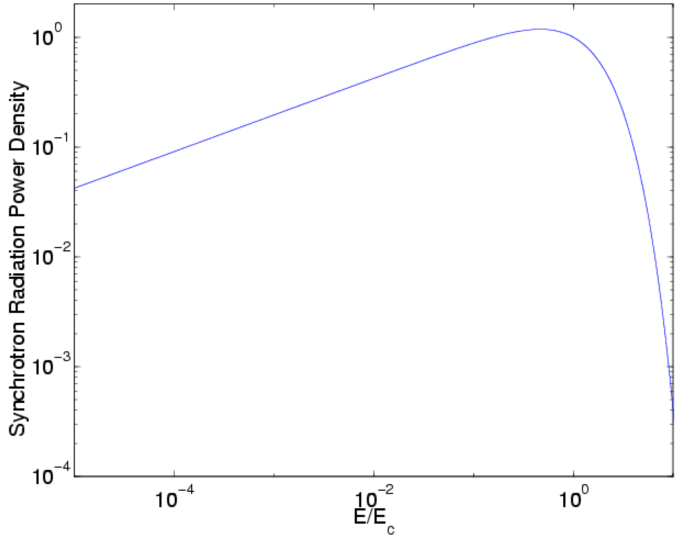
The power emitted by a particle is

$$P_{SR} = \frac{2}{3}\alpha\hbar c^2 \frac{\gamma^4}{\rho^2}$$

and the energy loss in one turn is

$$U_0 = \frac{4\pi}{3}\alpha\hbar c \frac{\gamma^4}{\rho^2}$$



# Radiation damping

Energy damping:

$$\alpha_D > 0 \qquad \alpha_D = -\frac{1}{2T_0}\frac{dU}{dE}\bigg|_{E_0} \quad \Rightarrow \quad \frac{dU}{dE}\bigg|_{E_0} < 0$$

Larger energy particles lose more energy

$$P_{SR} = \frac{2}{3}\alpha\hbar c^2 \frac{\gamma^4}{\rho^2}$$

• Typically, synchrotron radiation damping is very efficient in electron storage rings and negligible in proton machines.

• The damping time $1/\alpha_D$ (~ ms for e-, ~ 13 hours LHC at 7 TeV) is usually much larger than the period of the longitudinal oscillations $1/2\pi\Omega$ (~ μs). This implies that the damping term can be neglected when calculating the particle motion for $t << 1/\alpha_D$ :

$$\frac{d^2\Delta s}{dt^2} + \Omega^2 \Delta s = 0$$

**Harmonic oscillator equation**

# Quantum excitation - Longitudinally

The synchrotron radiation emitted as photons, the typical photon energy is

$$u_c = \hbar\omega_c = \frac{3}{2}\hbar c\frac{\gamma^3}{\rho}$$

The number of photons emitted is

$$N = \frac{4}{9}\alpha c\frac{\gamma}{\rho}$$

With a statistical uncertainty of $\sqrt{N}$

The equilibrium energy spread and bunch length is

$$\left(\frac{\sigma_e}{E}\right)^2 = 1.468\cdot 10^{-6}\frac{E^2}{J_\varepsilon \rho} \text{ and } \sigma_L = \frac{\alpha R}{f_0}\sigma_e$$